The LaplaceTransform

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Outline

- General form of the Laplace transform and its inverse
- Properties of Laplace transform
- Analyze and characterization of LTI systems using the Laplace transform

9.0 Introduction

- Fourier transform: using exponential signal e^{st} with s=jw to represent a arbitrary signals.
- Main reason: can be used to analyze unstable systems

9.1 The Laplace transform

- The general from:

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt$$

  - Observation 1: $s = \sigma + j\omega$;
  - When s=jw, reduce to Fourier transform
  - When s is not purely imaginary, we have

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$\Downarrow$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt$$

9.1 The Laplace transform

- Example 1: for signal $x(t)=e^{-at}u(t)$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a},\ a > 0$$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

$$= \frac{1}{(\sigma + a) + j\omega},\ \sigma + a > 0$$

$$= \frac{1}{s + a},\ \mathcal{Re}\{s\} > -a$$

9.1 The Laplace transform

- Example 2: for signal $x(t) = -e^{-at}u(-t)$

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= \frac{1}{s+a}$$

ROC: Region of convergence

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathcal{Re}\{s\} < -a$$

9.1 The Laplace transform

- A convenient way to display the ROC

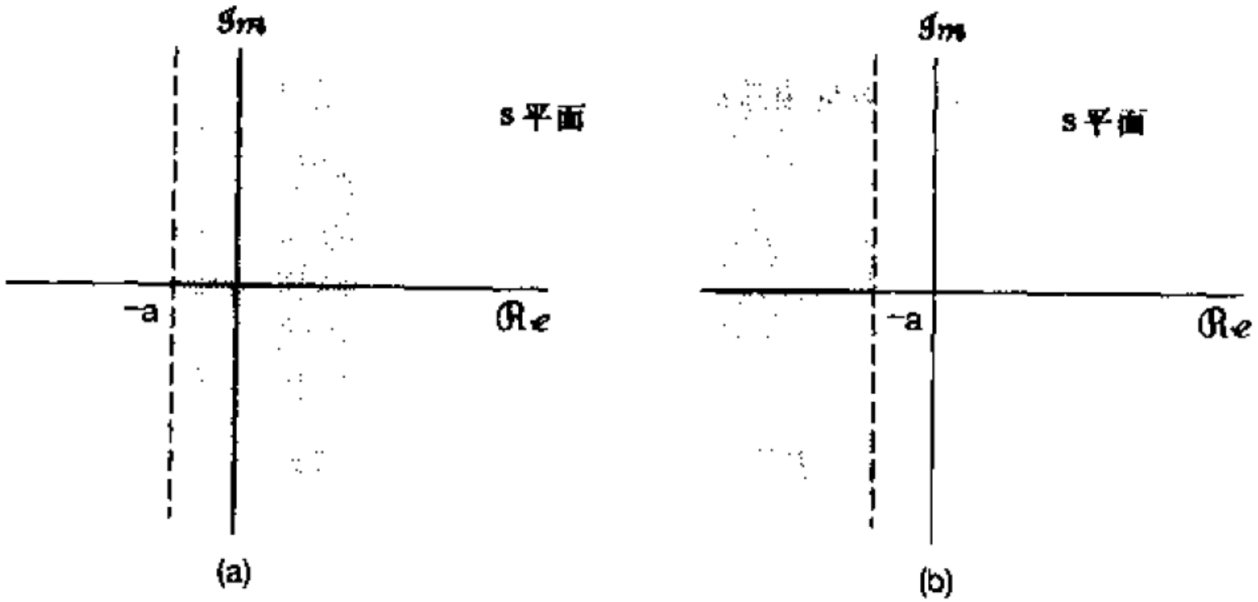


图 9.1 (a)例 9.1 的 ROC; (b)例 9.2 的 ROC

9.1 The Laplace transform

- Example 3: for signal

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

⇨

$$X(s) = \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}e^{-st}u(t)dt - 2\int_{-\infty}^{\infty} e^{-t}e^{-st}u(t)dt$$

$$= \frac{3}{s+2} - \frac{2}{s+1}$$

9.1 The Laplace transform

- We know

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{Re}\{s\} > -2$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s-1}{s^2+3s+2}, \quad \mathcal{Re}\{s\} > -1$$

9.1 The Laplace transform

- Example 4: for signal

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

$\Rightarrow$

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t}\right]u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1-3j)t}u(t)e^{-st}\,dt$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} e^{-(1+3j)t}u(t)e^{-st}\,dt$$

9.1 The Laplace transform

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{Re}\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1-3j)}, \quad \mathcal{Re}\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+(1+3j)}, \quad \mathcal{Re}\{s\} > -1$$

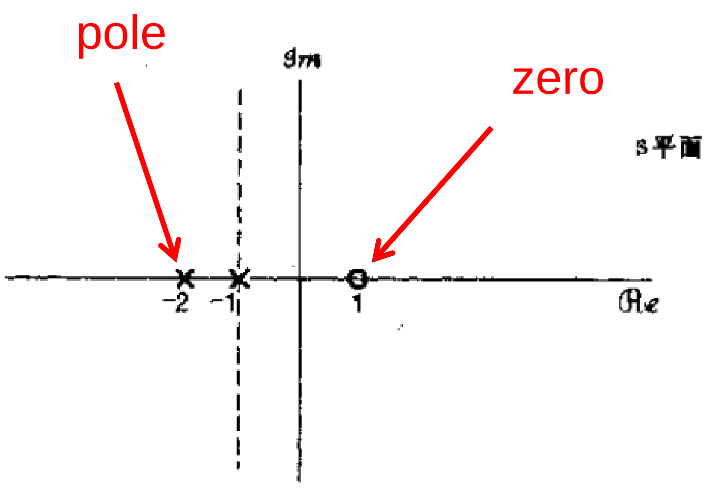
- Totally

$$\frac{1}{s+2} + \frac{1}{2}\left(\frac{1}{s+(1-3j)}\right) + \frac{1}{2}\left(\frac{1}{s+(1+3j)}\right), \quad \mathcal{Re}\{s\} > -1$$

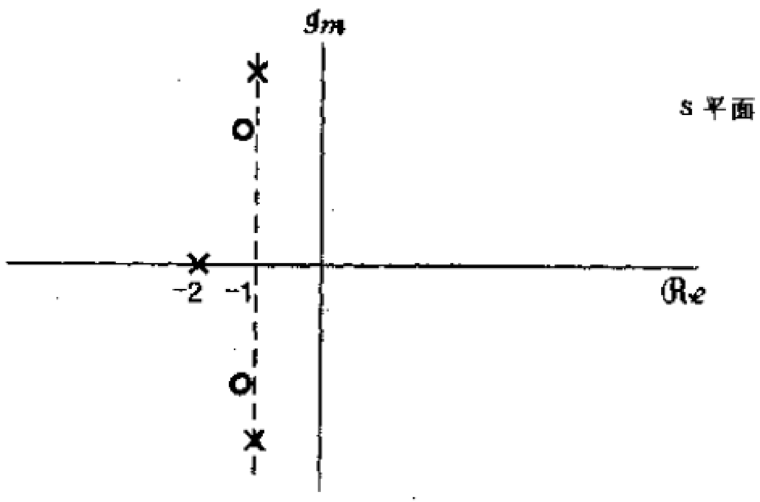
9.1 The Laplace transform

- Represent X(s) as:

$$X(s) = \frac{N(s)}{D(s)}.$$



(a)



(b)

9.1 The Laplace transform

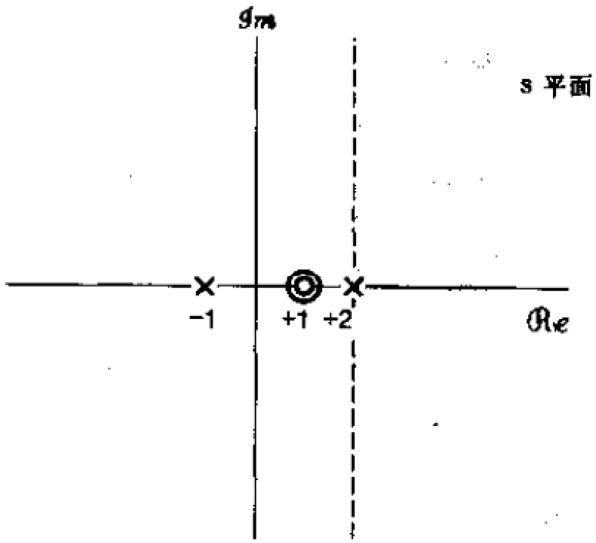
- Example 5: for signal

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

$$\mathscr{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = 1$$

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}, \quad \mathcal{Re}\{s\} > 2$$

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathcal{Re}\{s\} > 2$$



9.1 The Laplace transform

- Result: If the ROC of the Laplace transform does not include the jw-axis, then the Fourier transform does not converge.

9.2 The Region of Convergence for Laplace Transform

- Property 1: The ROC of X(s) consists of strips parallel to the jw-axis in the s-plane

$$\Rightarrow \quad \int_{-\infty}^{+\infty} |x(t)| e^{-\sigma t} dt < \infty$$

- Property 2: For rational Laplace transform, the ROC does not contain any poles

9.2 The Region of Convergence for Laplace Transform

- Property 3: If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

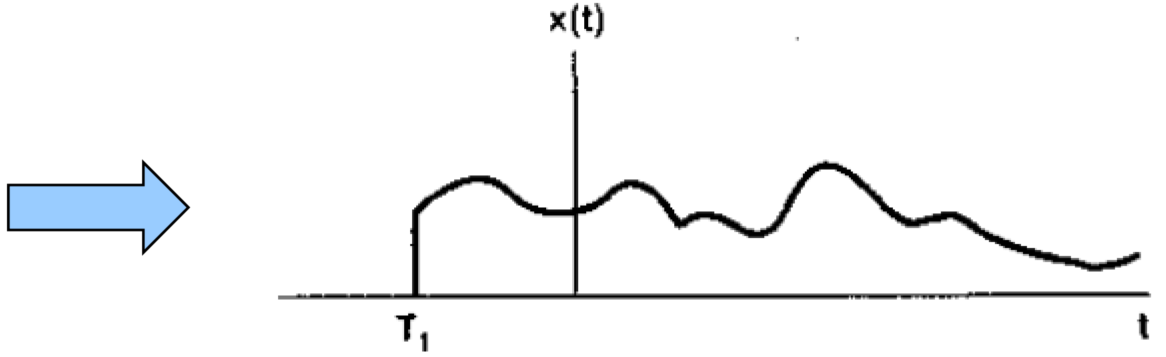
$$\int_{T_1}^{T_2} |x(t)|\,\mathrm{d}t < \infty$$

$$\begin{cases} \int_{T_1}^{T_2} |x(t)|\,\mathrm{e}^{-\sigma t}\mathrm{d}t < \mathrm{e}^{-\sigma T_1}\int_{T_1}^{T_2} |x(t)|\,\mathrm{d}t & \sigma>0 \\ \int_{T_1}^{T_2} |x(t)|\,\mathrm{e}^{-\sigma t}\mathrm{d}t < \mathrm{e}^{-\sigma T_2}\int_{T_1}^{T_2} |x(t)|\,\mathrm{d}t & \sigma<0 \end{cases}$$

9.2 The Region of Convergence for Laplace Transform

- Property 4: If x(t) is right sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} > \sigma_0$ will also be in the ROC.

Right-sided signal

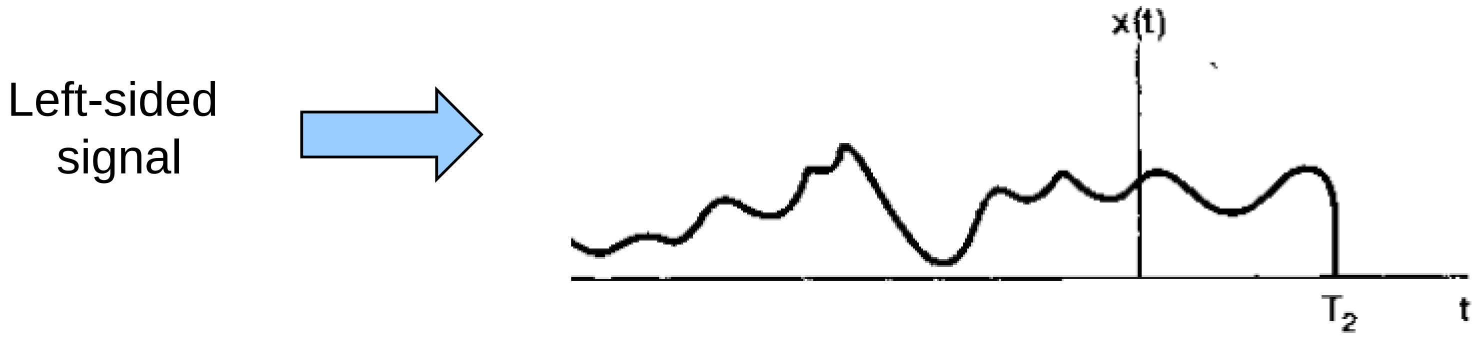


$$\int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt$$

$$\leqslant e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt \qquad \sigma_1 > \sigma_0;$$

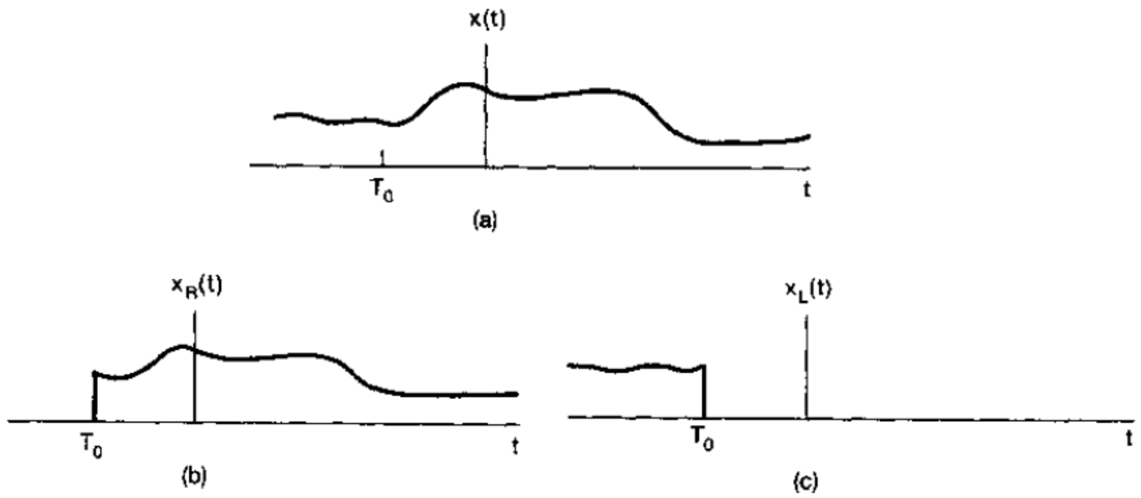
9.2 The Region of Convergence for Laplace Transform

- Property 5: If x(t) is left sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} < \sigma_0$ will also be in the ROC.



9.2 The Region of Convergence for Laplace Transform

- Property 6: If x(t) is two sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $\mathcal{Re}\{s\} = \sigma_0$.



9.2 The Region of Convergence for Laplace Transform

- Example 9.7: for signal $x(t) = e^{-b|t|}$,

$$\Rightarrow \quad x(t) = e^{-bt}u(t) + e^{+bt}u(-t)$$

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b}, \quad \mathcal{Re}\{s\} > -b$$

$$e^{+bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b}, \quad \mathcal{Re}\{s\} < +b$$

$$e^{-b|t|} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2-b^2}, \quad -b < \mathcal{Re}\{s\} < +b$$

9.2 The Region of Convergence for Laplace Transform

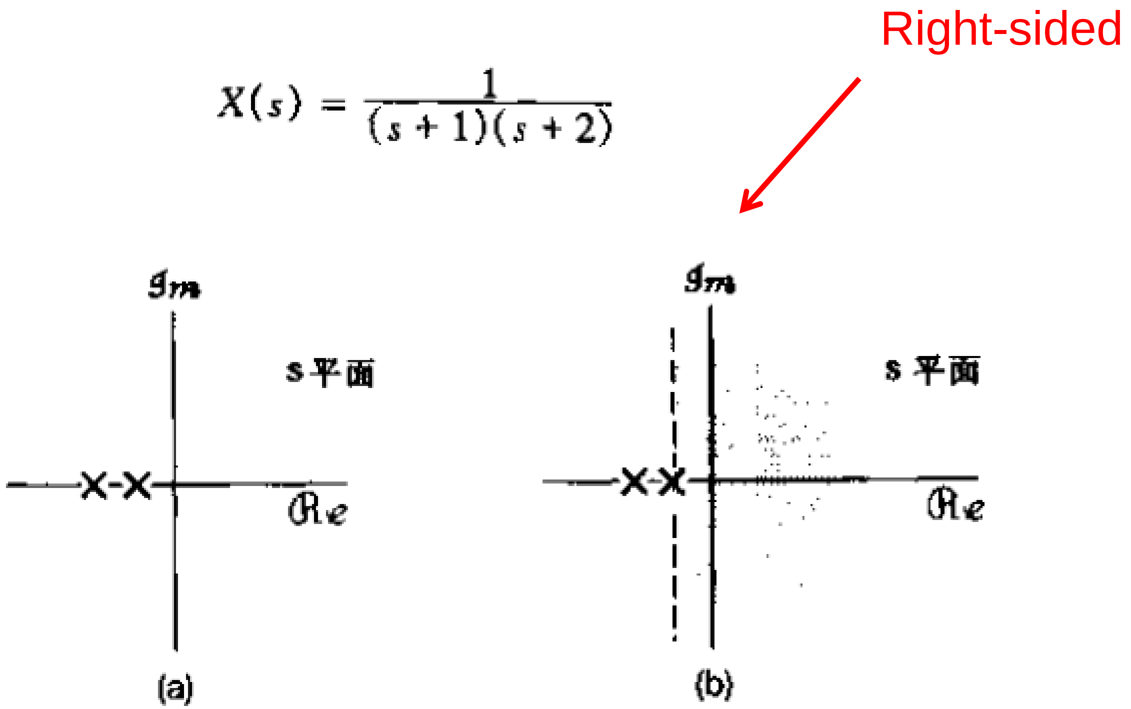
- Property 7: If the Laplace transform X(s) of x(t) is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of X(s) are contained in the ROC.

- Property 8: If the Laplace transform X(s) of x(t) is rational, then if x(t) is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If x(t) is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

9.2 The Region of Convergence for Laplace Transform

- Example 9.8: for signal having a Laplace transform given by

$$X(s) = \frac{1}{(s+1)(s+2)}$$



9.2 The Region of Convergence for Laplace Transform

- Example 9.8: for signal having a Laplace transform given by

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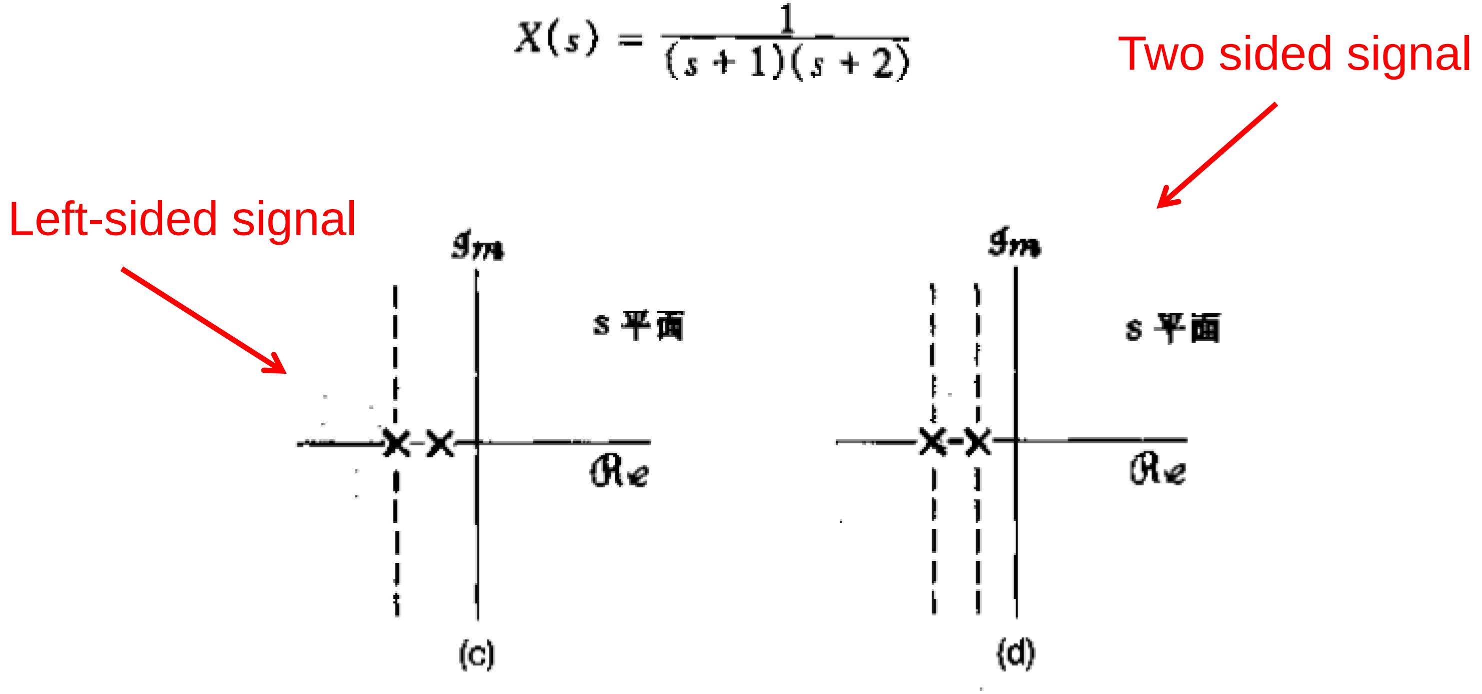


9.3 The Inverse Laplace Transform

- The Laplace transform of a signal x(t) is

$$X(\sigma + j\omega) = \mathscr{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt$$

⇓

$$x(t)e^{-\sigma t} = \mathscr{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

⇓

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega \Rightarrow \boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds}$$

9.3 The Inverse Laplace Transform

- Assuming no multiple-order poles, and that the order of the denominator polynomial is greater than the order of the numerator polynomial, we can expand X(s) in a form of

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

9.3 The Inverse Laplace Transform

- Then, based on the ROC of X(s), the inverse Laplace transform of each of these terms can be determined.

$$A_i/(s+a_i) \rightarrow A_i e^{-a_i t} u(t)$$

$$A_i/(s+a_i) \rightarrow -A_i e^{-a_i t} u(-t)$$

9.3 The Inverse Laplace Transform

- Example 9.9: Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \mathcal{Re}\{s\} > -1$$

$$X(s) = \frac{1}{(s+1)(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

9.3 The Inverse Laplace Transform

- Example 9.9: Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \mathcal{Re}\{s\} > -1$$

⇨

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{Re}\{s\} > -2$$

$$[e^{-t} - e^{-2t}]u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathcal{Re}\{s\} > -1$$

9.3 The Inverse Laplace Transform

- Example 9.10: Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{Re}\{s\} < -2$$

$$-e^{-t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathcal{Re}\{s\} < -1$$

$$-e^{-2t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathcal{Re}\{s\} < -2$$

$$x(t) = [-e^{-t} + e^{-2t}]u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad \mathcal{Re}\{s\} < -2$$

9.3 The Inverse Laplace Transform

- Example 9.11: Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad -2 < \mathcal{Re}\{s\} < -1$$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+1)(s+2)}, \quad -2 < \mathcal{Re}\{s\} < -1$$

9.4 Properties of the Laplace Transform

- Linearity:

$$x_1(t) \overset{\mathscr{L}}{\longleftrightarrow} X_1(s),\quad \text{ROC 为 } R_1$$

$$x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} X_2(s),\quad \text{ROC 为 } R_2$$

$$\boxed{a x_1(t) + b x_2(t) \overset{\mathscr{L}}{\longleftrightarrow} a X_1(s) + b X_2(s),\quad \text{ROC 包括 } R_1 \cap R_2}$$

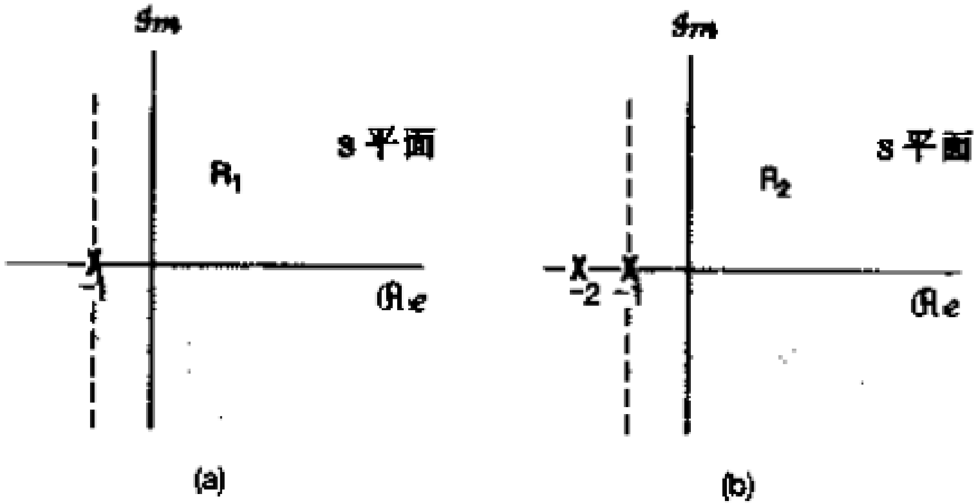
9.4 Properties of the Laplace Transform

- Example 9.13:

$$x(t) = x_1(t) - x_2(t)$$

$$X_1(s) = \frac{1}{s+1}, \quad \Re e\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \Re e\{s\} > -1$$

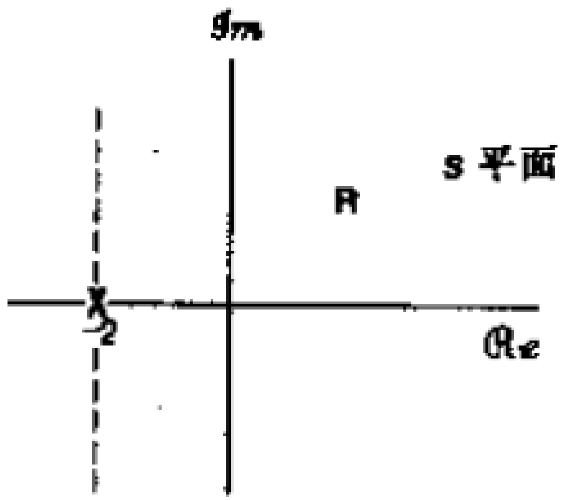


9.4 Properties of the Laplace Transform

- Example 9.13:

$$x(t) = x_1(t) - x_2(t)$$

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$



(c)

9.4 Properties of the Laplace Transform

- Shifting in the time domain & S domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad \text{ROC} = R$$

$$\boxed{x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s), \qquad \text{ROC} = R}$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \qquad \text{ROC} = R$$

$$\boxed{e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0), \qquad \text{ROC} = R + \mathcal{Re}\{s_0\}}$$

$$\Rightarrow \quad e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - j\omega_0), \qquad \text{ROC} = R$$

9.4 Properties of the Laplace Transform

- Time scaling

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \qquad \text{ROC} = R$$

$$\boxed{x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right), \qquad \text{ROC} R_1 = \frac{R}{a}}$$

$\Rightarrow$ $$x(-t) \overset{\mathcal{L}}{\longleftrightarrow} X(-s), \qquad \text{ROC} = -R$$

9.4 Properties of the Laplace Transform

- Conjugation

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s) \qquad \text{ROC} = R$$

$$x^*(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X^*(s^*), \qquad \text{ROC} = R$$

$\Rightarrow$ $$X(s) = X^*(s^*)$$

9.4 Properties of the Laplace Transform

- Convolution Property

$$x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s), \qquad \text{ROC} = R_1$$

$$x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s), \qquad \text{ROC} = R_2$$

$$\boxed{x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)X_2(s), \qquad \text{ROC 包括 } R_1 \cap R_2}$$

$$X_1(s) = \frac{s+1}{s+2}, \qquad \mathcal{Re}\{s\} > -2$$

$$X_2(s) = \frac{s+2}{s+1}, \qquad \mathcal{Re}\{s\} > -1$$

9.4 Properties of the Laplace Transform

- Differentiation in the time domain

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{ROC} = R$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \overset{\mathcal{L}}{\longleftrightarrow} sX(s), \qquad \text{ROC 包括 } R$$

9.4 Properties of the Laplace Transform

- Differentiation in the S-domain

$$x(t) \overset{\mathscr{L}}{\longleftrightarrow} X(s), \qquad \text{ROC} = R$$

$$-tx(t) \overset{\mathscr{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad \text{ROC} = R$$

9.4 Properties of the Laplace Transform

- Example 9.14: find the Laplace transform of

$$x(t) = te^{-at}u(t)$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \quad \mathcal{R}e\{s\} > -a$$

$$te^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \mathcal{R}e\{s\} > -a$$

$$\frac{t^2}{2}e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{(s+a)^3}, \quad \mathcal{R}e\{s\} > -a$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{(s+a)^n}, \quad \mathcal{R}e\{s\} > -a$$

9.4 Properties of the Laplace Transform

- Integration in the time domain

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{ROC} = R$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s), \quad \text{ROC 包括 } R \cap \{\mathcal{Re}\{s\} > 0\}$$

Proof:

$$\int_{-\infty}^{t} x(\tau)\,d\tau = u(t) * x(t)$$

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}, \qquad \mathcal{Re}\{s\} > 0$$

$$u(t) * x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)$$

9.4 Properties of the Laplace Transform

- The initial and final value theorems
  - Initial-value theorem

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

  - Final-value theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

  - Condition: (1) signal x(t) =0 for t<0 and that x(t) contains no impulses or high order singularities at the origin. (2) signal x(t) =0 for t<0 and that x(t) has a finite limit as t approaches infinity.

9.5 Some Laplace transform pairs

表 9.2 基本函数的拉普拉斯变换

变换对	信号	变换	ROC
1	$\delta(t)$	1	全部 s
2	$u(t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\}>0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathcal{Re}\{s\}<0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\}>0$
5	$\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\}<0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\}>-\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\mathcal{Re}\{s\}<-\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\}>-\alpha$

9.5 Some Laplace transform pairs

9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\mathcal{Re}\{s\}<-\alpha$
10	$\delta(t-T)$	e^{-sT}	全部 s
11	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\}>0$
12	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathcal{Re}\{s\}>0$
13	$[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\}>-\alpha$
14	$[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\mathcal{Re}\{s\}>-\alpha$
15	$u_n(t)=\frac{d^n\delta(t)}{dt^n}$	s^n	全部 s
16	$u_{-n}(t)=\underbrace{u(t)*\cdots*u(t)}_{n次}$	$\frac{1}{s^n}$	$\mathcal{Re}\{s\}>0$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Using the convolution property, we have

$$Y(s) = H(s)X(s)$$

  - H(s) is referred to as the system function & transfer function

9.6 Analysis and characterization of LTI system using the Laplace transform

- 1. Causality

  - The ROC associated with the system function for a causal system is a right-half plane.

  - If H(s) is rational, then we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane.

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.17: consider a system with unit impulse response

$$h(t) = e^{-t}u(t)$$

$$H(s) = \frac{1}{s+1}, \qquad \mathcal{Re}\{s\} > -1$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.18: consider a system with unit impulse response

$$h(t) = e^{-|t|}$$

$$H(s) = \frac{-2}{s^2 - 1}, \qquad -1 < \Re\{s\} < +1$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.19: consider a system with system function given as

$$H(s) = \frac{e^s}{s+1}, \qquad \mathcal{Re}\{s\} > -1$$

$$\Rightarrow \quad e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \qquad \mathcal{Re}\{s\} > -1$$

$$e^{-(t+1)}u(t+1) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^s}{s+1}, \qquad \mathcal{Re}\{s\} > -1$$

$$h(t) = e^{-(t+1)}u(t+1)$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- 2. Stability

  - An LTI system is stable if and only if the ROC of its system function H(s) includes the entire jw-axis.

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.20: consider a system with system function given as

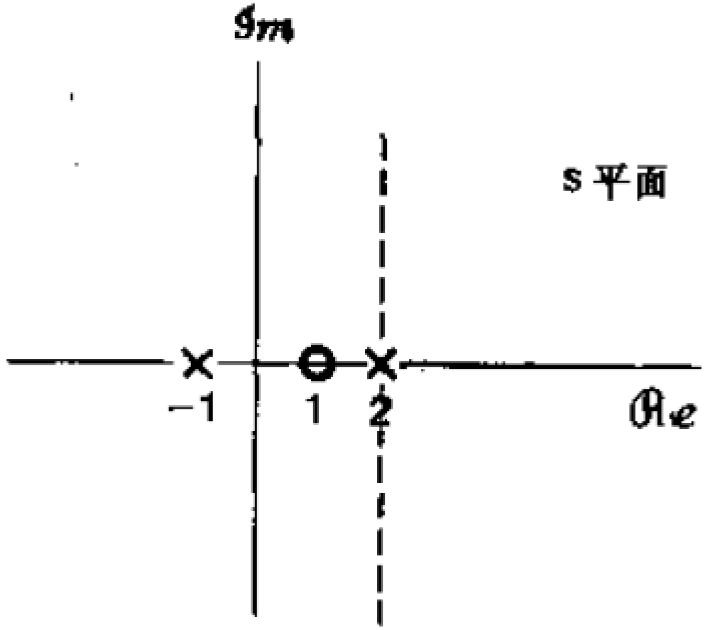
$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

Determine the stability.

9.6 Analysis and characterization of LTI system using the Laplace transform

- case 1: unstable system

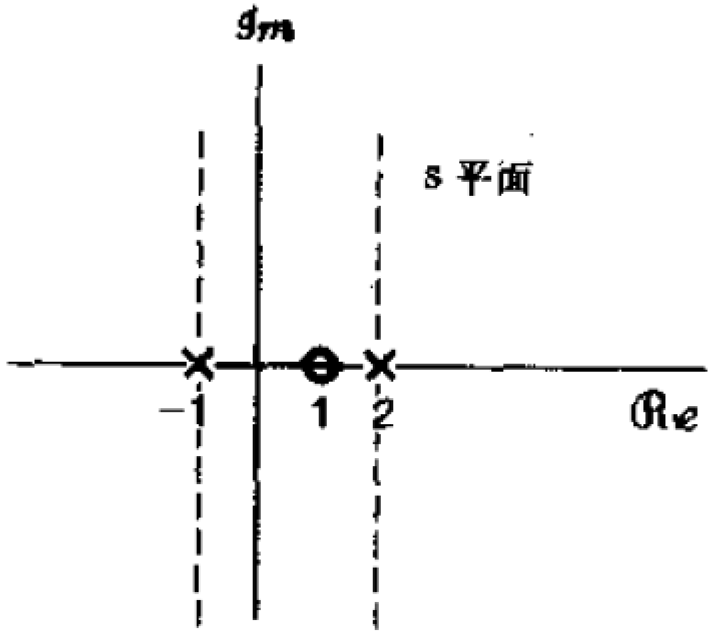
$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$$



9.6 Analysis and characterization of LTI system using the Laplace transform

- case 2: stable system

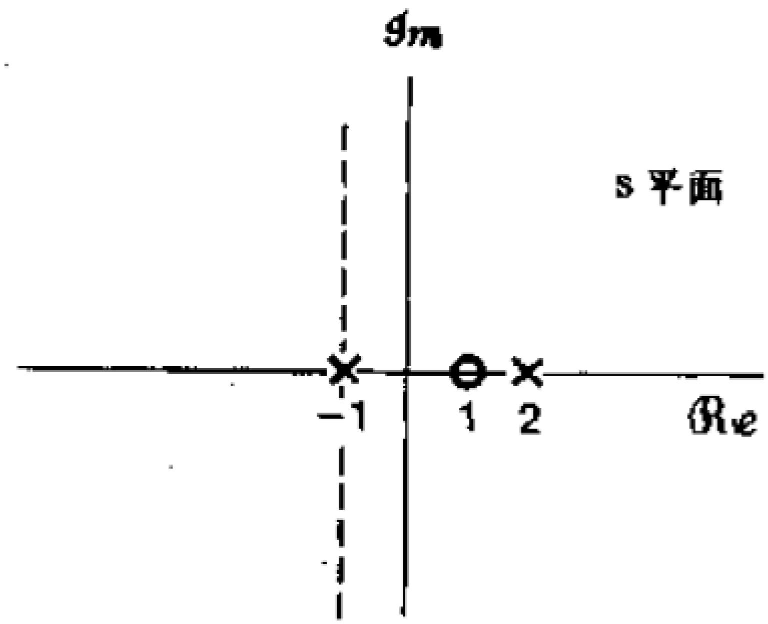
$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$$



9.6 Analysis and characterization of LTI system using the Laplace transform

- case 3: unstable system

$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)$$



9.6 Analysis and characterization of LTI system using the Laplace transform

- 2. Stability: for causal LTI system

  - A causal system with rational system function H(s) is stable if and only if all of the poles of H(s) lie in the left-half of the s-plane, i.e., all of the poles have negative real parts.

9.6 Analysis and characterization of LTI system using the Laplace transform

- 3. LTI system characterized by linear constant-coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

$$H(s) = \frac{\left\{\sum_{k=0}^{M} b_k s^k\right\}}{\left\{\sum_{k=0}^{N} a_k s^k\right\}}$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- 3. LTI system characterized by linear constant-coefficient differential equations
  - zeros at solutions of

$$\sum_{k=0}^{M} b_k s^k = 0$$

  - Poles at the solutions of

$$\sum_{k=0}^{N} a_k s^k = 0$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.25: the input to an LTI system is

$$x(t) = e^{-3t}u(t)$$

and the output is

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

$$\Rightarrow \quad X(s) = \frac{1}{s+3}, \quad \mathcal{Re}\{s\} > -3$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \mathcal{Re}\{s\} > -1$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Further, we have

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s+3}{(s+1)(s+2)}=\frac{s+3}{s^2+3s+2}$$

  - Case 1: ROC is to the left of the pole at s = -2
  - Case 2: ROC is between the poles at -2 and -1
  - Case 3: ROC is to the right of the pole at s = -1

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t)$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.26: Suppose that we are given the following information of an LTI system:
  - The system is causal
  - The system function is rational and has only two poles, at s = -2 and s = 4;
  - If x(t) = 1, then y(t) = 0;
  - The value of the impulse response at t= 0+ is 4;

Determine the system function.

9.6 Analysis and characterization of LTI system using the Laplace transform

- Solution:
  - The system is causal but is unstable since it has pole at s = 4 with positive real part. The system function is of the form

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8}$$

  - y(t) to the input x(t) = 1= exp(0・t) equal H(0)・exp(0・t) = H(0), we have p(0) = 0, which further means p(s) has a root at s = 0 and is of the form

$$p(s) = sq(s)$$

  - Finally, from fact 4 and the initial theorem, we see that

$$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = 4$$

implying q(s) = 4.

$$H(s) = \frac{4s}{(s+2)(s-4)}$$

9.6 Analysis and characterization of LTI system using the Laplace transform

- Example 9.27: Consider a stable and casual system with impulse response h(t) and system function H(s). H(s) is rational, contains a pole at s = -2 and does not have zero at the origin. Determine whether the following statement is true, whether we can definitely say that is false, or whether there is insufficient information to determine the turth.
  - (a) $\mathcal{F}\{h(t)e^{3t}\}$ converges
  - (b) $\int_{-\infty}^{+\infty} h(t)dt = 0$
  - (c) t · h(t) is the impulse response of a causal and stable system

9.6 Analysis and characterization of LTI system using the Laplace transform

- (d) $dh(t)/dt$ contains at least on pole in its Laplace transform
- (e) h(t) has finite duration
- (f) H(s) = H(-s)
- (g) $\lim_{s \to \infty} H(s) = 2$,

9.6 Analysis and characterization of LTI system using the Laplace transform

- Solution:
  - (a) false, since the above Fourier transform corresponds to the value of the Laplace transform of h(t) at s = -3. If this is converges, it implies that s = -3 is in the ROC. A causal and stable system must always have its ROC to the right of the all of its poles. However, s = -3 is not to the right of the pole at s = -2.
  - (b) false, since it is equivalent to H(0) = 0. This contradicts the fact that H(s) does not have a zero at the origin.
  - (c) true. The Laplace transform th(t) has the same ROC as that of H(s). This ROC contains jw axis, thus is stable.

9.6 Analysis and characterization of LTI system using the Laplace transform

- Solution:
  - (d) true, as it has a Laplace transform sH(s), which contains a pole at s = -2.
  - (e) false, If h(t) is of finite duration, then if its Laplace transform has any points in its ROC, the ROC must be the entire s-plane, which is not consistent with the fact that H(s) having a pole at s = -2.
  - (f) false, if it is true, H(s) must have a pole at s = 2. This is not consistent with the fact that all the poles of a causal and stable system must be in the left half of the s-plane.
  - (g) the statement can not be determined with the information given. The statement requires that the degree of the numerator and denominator of H(s) be equal. We have insufficient information about H(s) to determine if is the case.