The Discrete-time Fourier Transform

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Outline

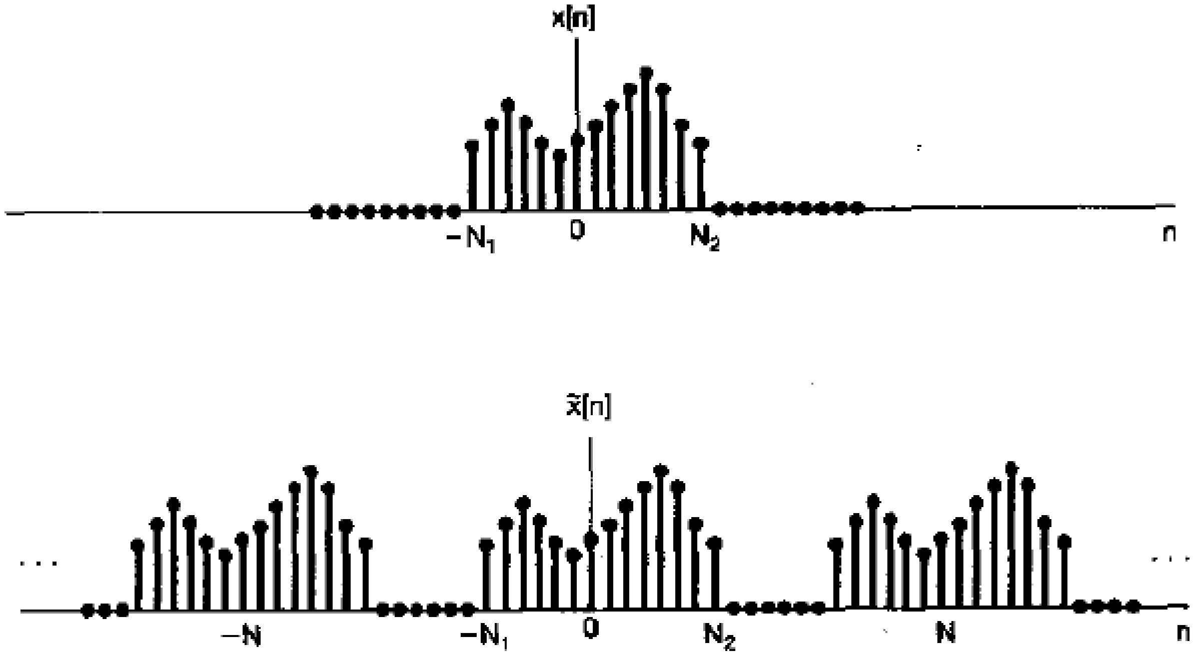
- Representation of Aperiodic signals: The discrete-time Fourier Transform
- The Fourier transform for periodic signals
- Properties of the discrete-time Fourier transform
- The convolution/multiplication property
- System characterization by linear constant-coefficient difference equations

5.1 The discrete-time Fourier transform

- The difference of the Fourier series representation between continuous-time case and discrete-time case
  - The continuous-time signal needs to represented by infinite number of exponential signals
  - The discrete-time signal needs to be represented by finite number exponential signals

5.1 The discrete-time Fourier transform

- Extend non-periodic signal to periodic signal



5.1 The discrete-time Fourier transform

- Fourier series representation

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk(2\pi/N)n}$$

- Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

- We have $a_k = \frac{1}{N}X(e^{jk\omega_0}) \qquad \omega_0 = 2\pi/N$

5.1 The discrete-time Fourier transform

- Then we express

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N\rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

- When $N \to \infty$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

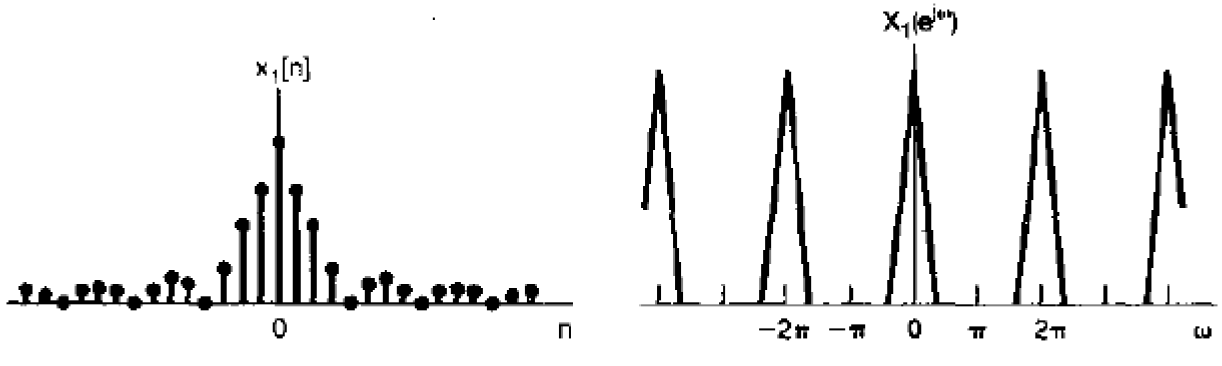
5.1 The discrete-time Fourier transform

- The Fourier transform is

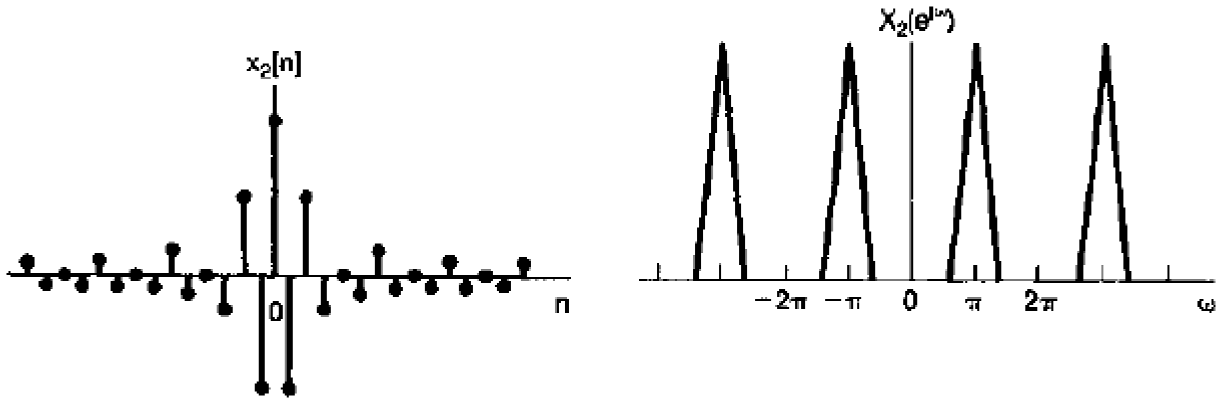
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

5.1 The discrete-time Fourier transform



(a) (b)



(c) (d)

5.1 The discrete-time Fourier transform

- Example 1: consider a signal

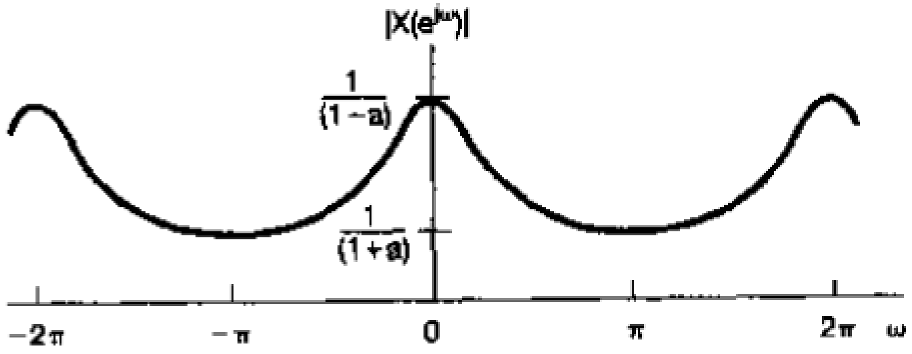
$$x[n] = a^n u[n], \qquad |a| < 1$$

Determine the Fourier transform.

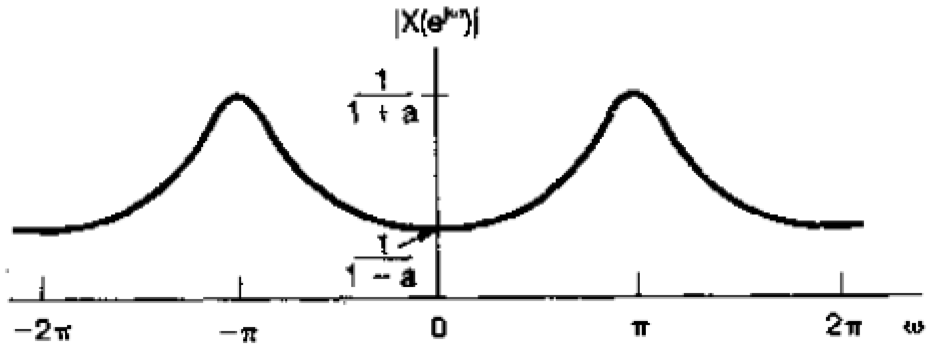
- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

5.1 The discrete-time Fourier transform



$a > 0$



$a < 0$

5.1 The discrete-time Fourier transform

- Example 2: consider a signal

$$x[n] = a^{|n|}, \qquad 0<a<1$$

Determine the Fourier transform.

- Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

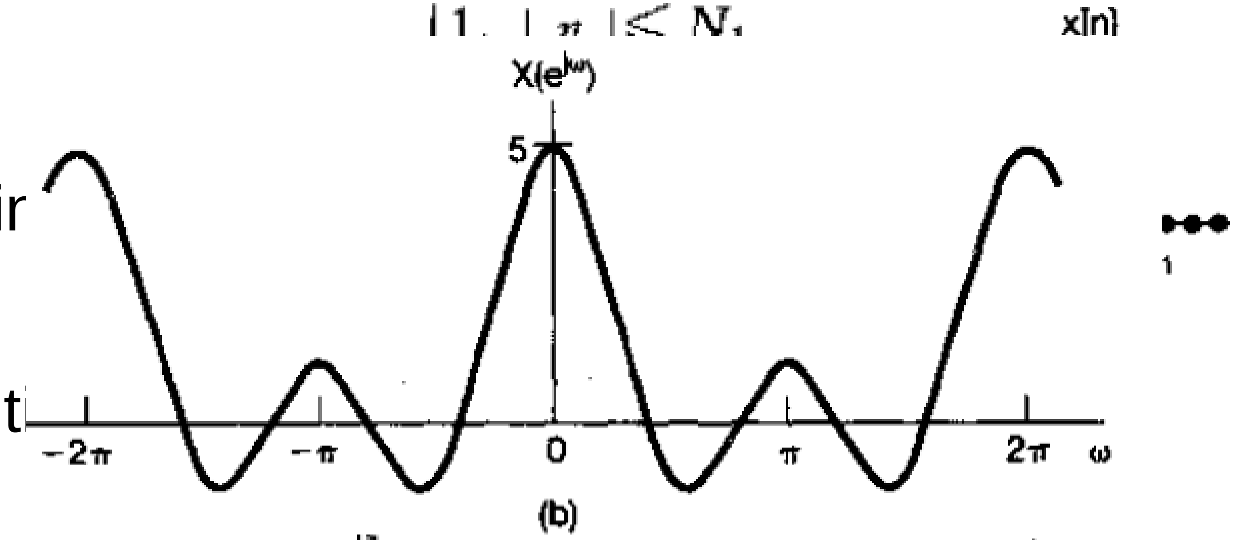
$$X(e^{j\omega}) = \sum_{n=0}^{\infty} (a e^{-j\omega})^{n} + \sum_{m=1}^{\infty} (a e^{j\omega})^{m}$$

$$X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

5.1 The discrete-time Fourier transform

- Example 3: consider a rectangular pulse

Determir



- Soluti

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{\sin\omega\left(N_1+\frac{1}{2}\right)}{\sin(\omega/2)}$$

5.1 The discrete-time Fourier transform

- Example 4: consider a signal

$$x[n] = \delta[n]$$

Determine the Fourier transform.

- Solution:

$$X(e^{j\omega}) = 1$$

5.1 The discrete-time Fourier transform

- The Convergence issue:
  - Finite duration: always convergent
  - infnite duration:

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

or

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$

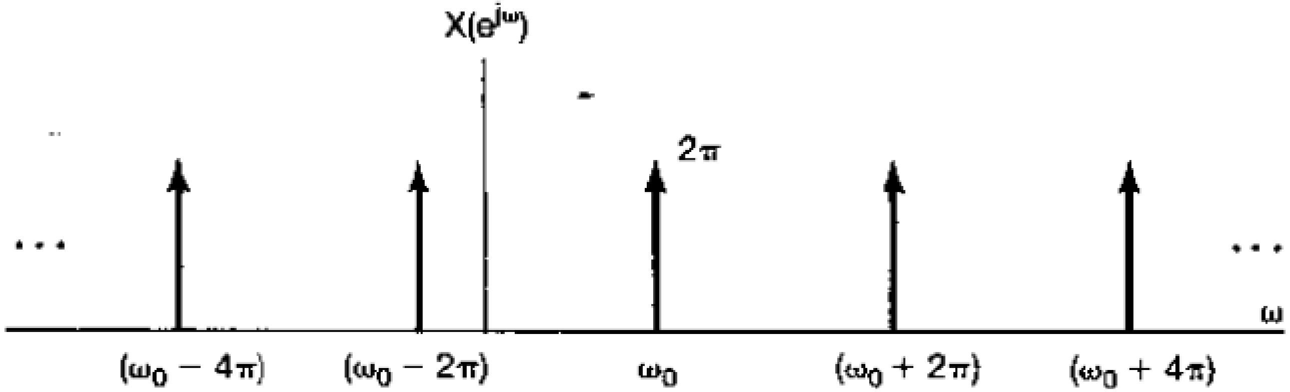
5.2 The Fourier transform for periodic signals

- The Fourier transform of exponential signal

$$x[n] = e^{j\omega_0 n}$$

- We have

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



5.2 The Fourier transform for periodic signals

- Why?

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega$$

$$= e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}$$

5.2 The Fourier transform for periodic signals

- The Fourier transform of periodic signal

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

- We have

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

5.2 The Fourier transform for periodic signals

- Why?

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \cdots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

- W_0 are:

$$\omega_0 = 0, 2\pi/N, \ 4\pi/N, \ \cdots, \ \ (N-1)2\pi/N$$

5.2 The Fourier transform for periodic signals

- Example 2.1: consider a periodic signal

$$x[n] = \cos\omega_0 n \qquad \omega_0 = \frac{2\pi}{5}$$

Determine the Fourier transform

- Solution:

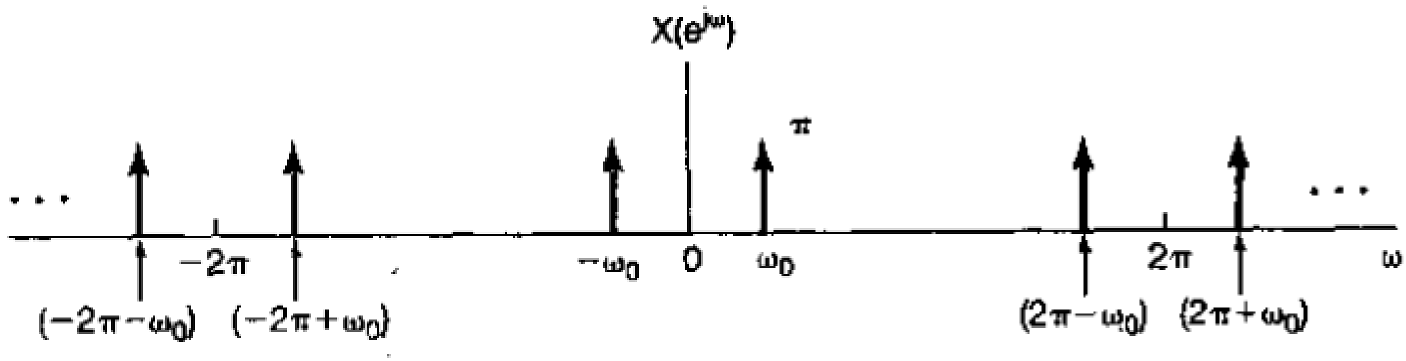
$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty}\pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty}\pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

5.2 The Fourier transform for periodic signals

- Solution:

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \quad -\pi \leqslant \omega < \pi$$



5.2 The Fourier transform for periodic signals

- Example 2.2: consider a periodic signal

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

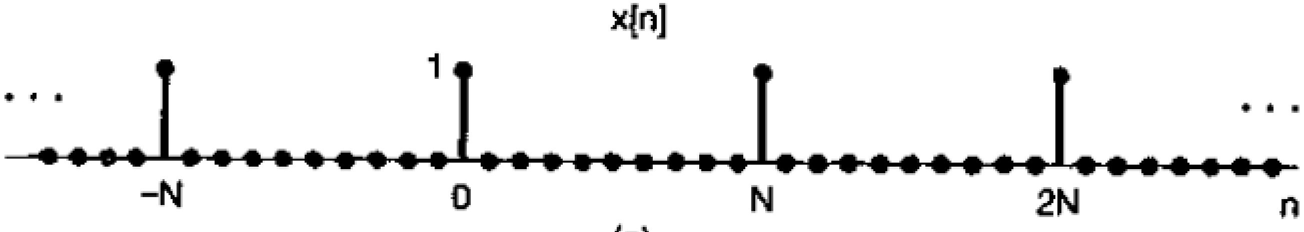
Determine the Fourier transform

- Solution: $$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

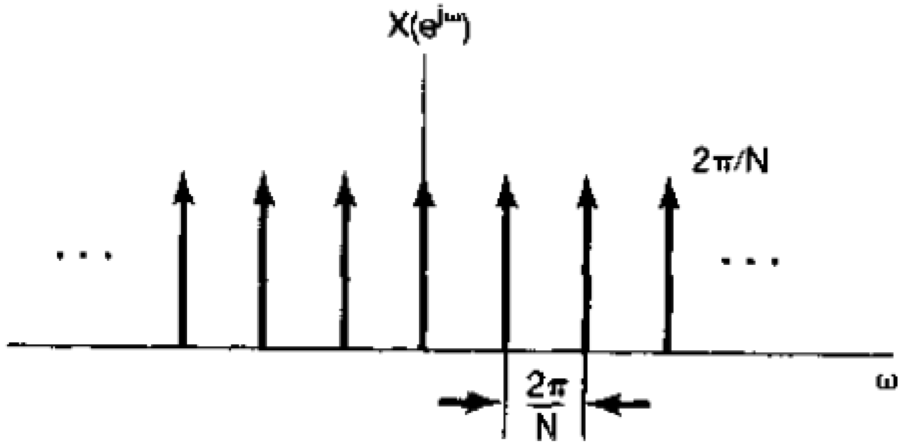
$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

5.2 The Fourier transform for periodic signals

- Solution:



(a)



(b)

5.2 The Fourier transform for periodic signal

- Example 2: Consider the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

- Determine its Fourier transform
- Solution:

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

5.3 Properties

- Linearity:

$$a x_1[n] + b x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} a X_1(e^{j\omega}) + b X_2(e^{j\omega})$$

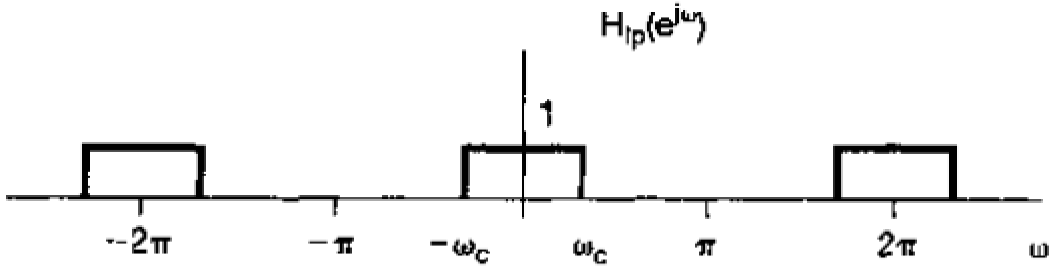
- Time shifting:

$$x[n - n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega - \omega_0)})$$

5.3 Properties

- Example 3.1: Consider the frequency response $H_{lp}(e^{j\omega})$ of the lowpass filter with cutoff frequency ω_c



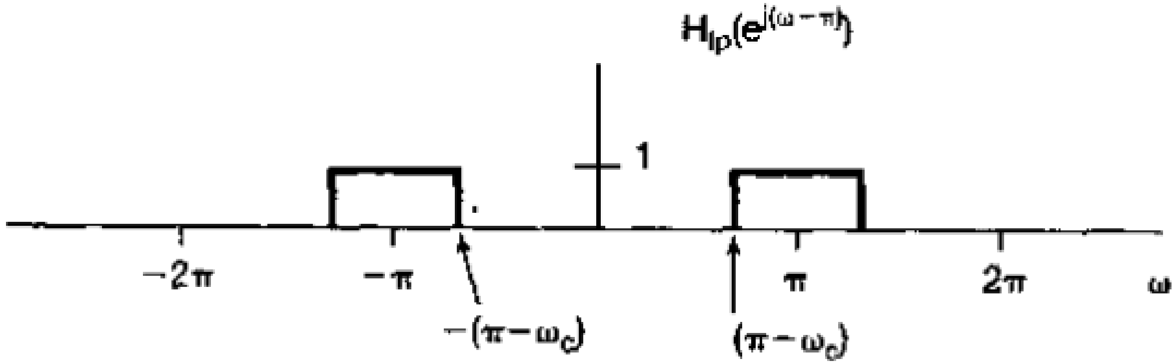
(a)

An ideal highpass filter is

$$H_{hp}(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})$$

with cutoff frequency $\pi - \omega_c$

5.3 Properties



(b)

- The frequency shifting property implies

$$h_{hp}[n] = e^{j\pi n} h_{lp}[n]$$
$$= (-1)^n h_{lp}[n]$$

5.3 Properties

- Conjugation &conjugate symmetry

$$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

  - If x[n] is real

$$X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$\mathcal{E}v\{X[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\,\mathcal{I}m\{X(e^{j\omega})\}$$

5.3 Properties

- Differencing and accumulation

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

5.3 Properties

- Example 3.2: use Fourier transform of delta function to derive the Fourier transform of unit step function

$$g[n] = \delta[n] \overset{\mathcal{F}}{\longleftrightarrow} G(e^{j\omega}) = 1$$

$$x[n] = \sum_{m=-\infty}^{n} g[m]$$

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})}G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

5.3 Properties

- Time reversal

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Proof:

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[-n]e^{-j\omega n}$$

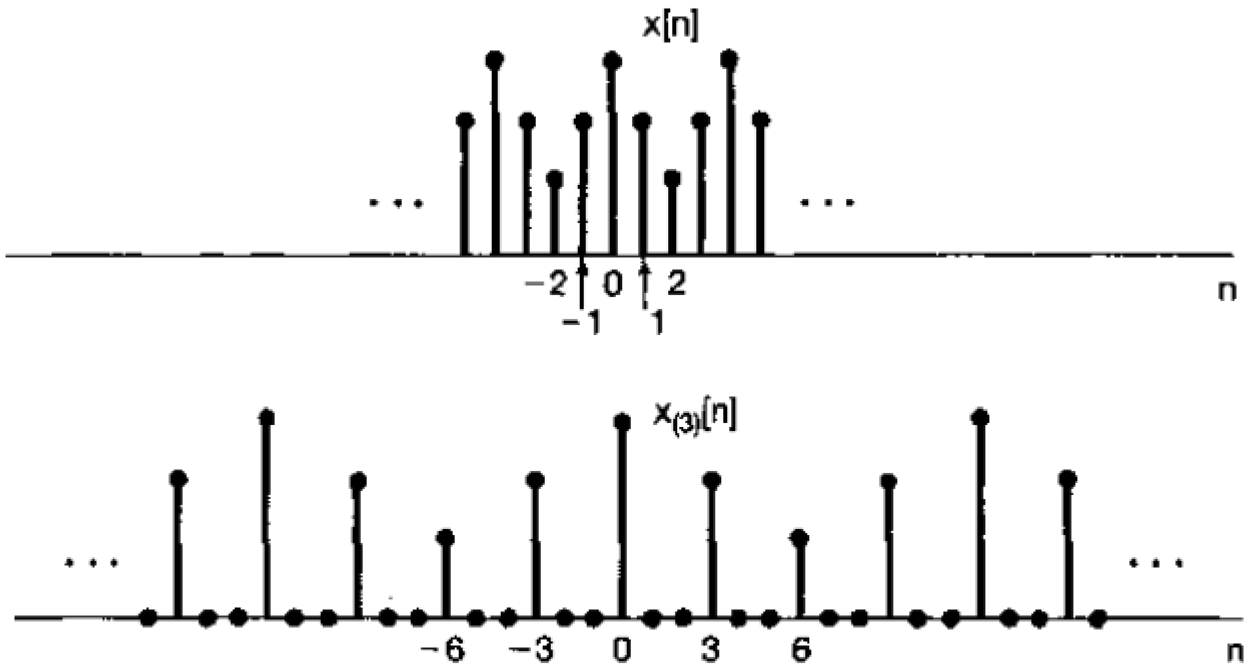
Replace m by -n

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[m]e^{-j(-\omega)m} = X(e^{-j\omega})$$

5.3 Properties

- Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{If n is a multiple of k} \\ 0, & \text{If n is not a multiple of k} \end{cases}$$



5.3 Properties

- Then we have

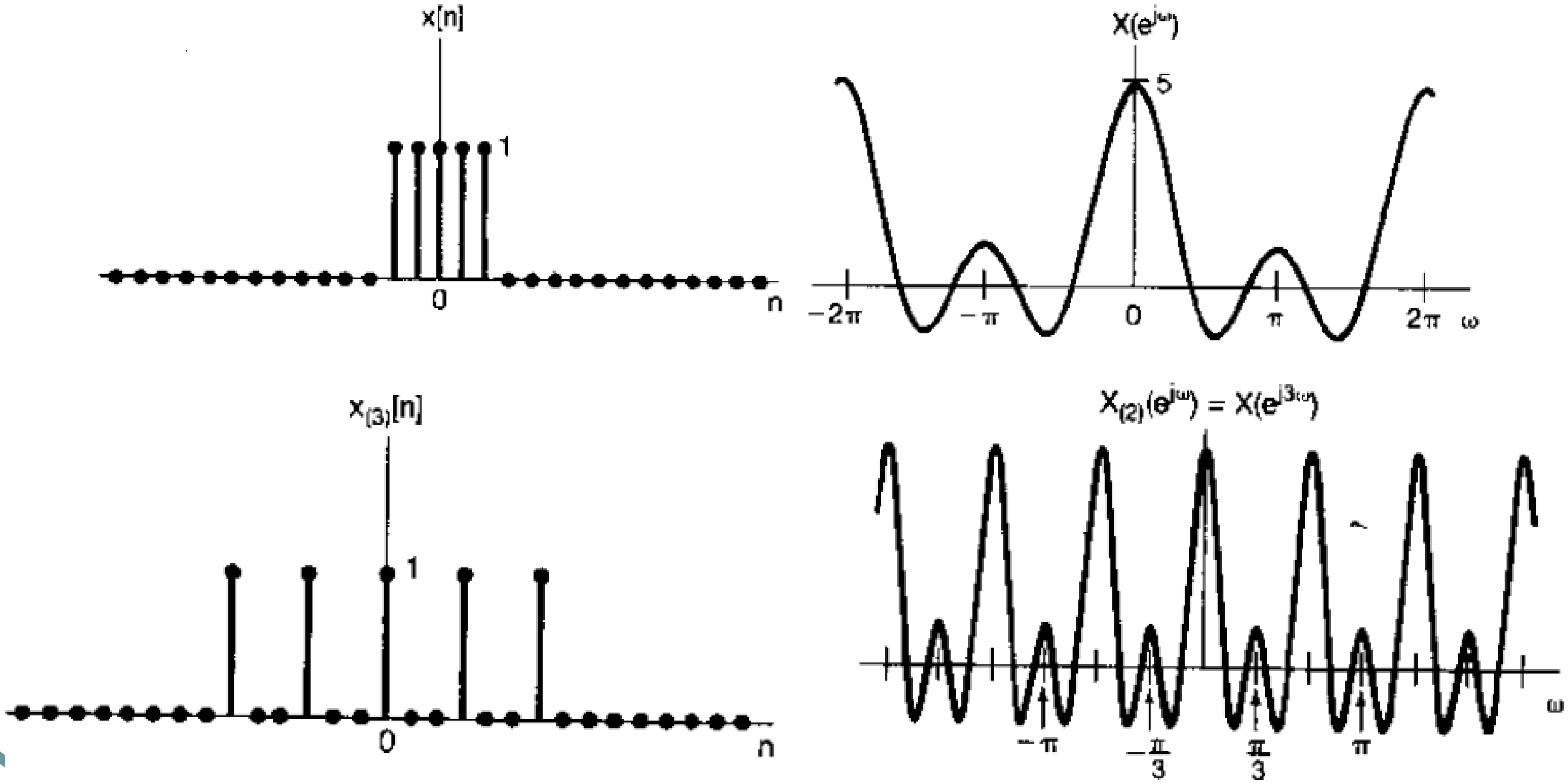
$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

$$= \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$\boxed{x_{(k)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega})}$$

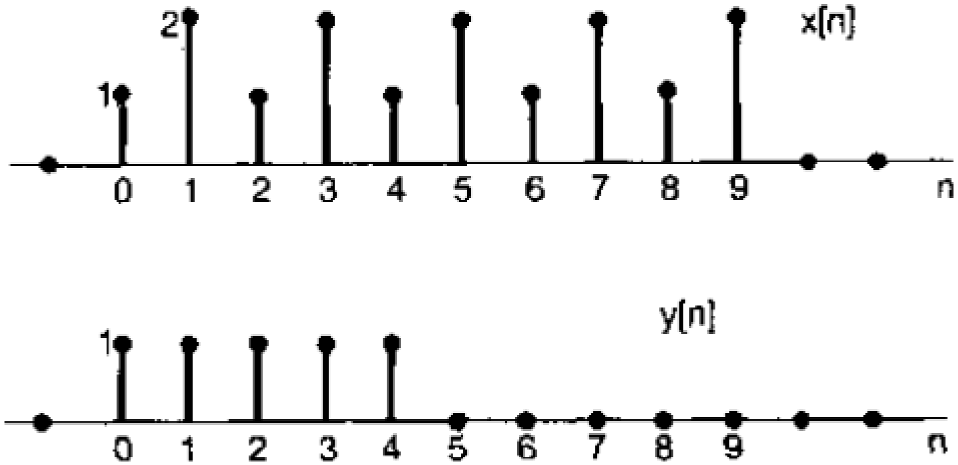
5.3 Properties

- When k>1, the signal is spread out, while its Fourier transform is compressed.



5.3 Properties

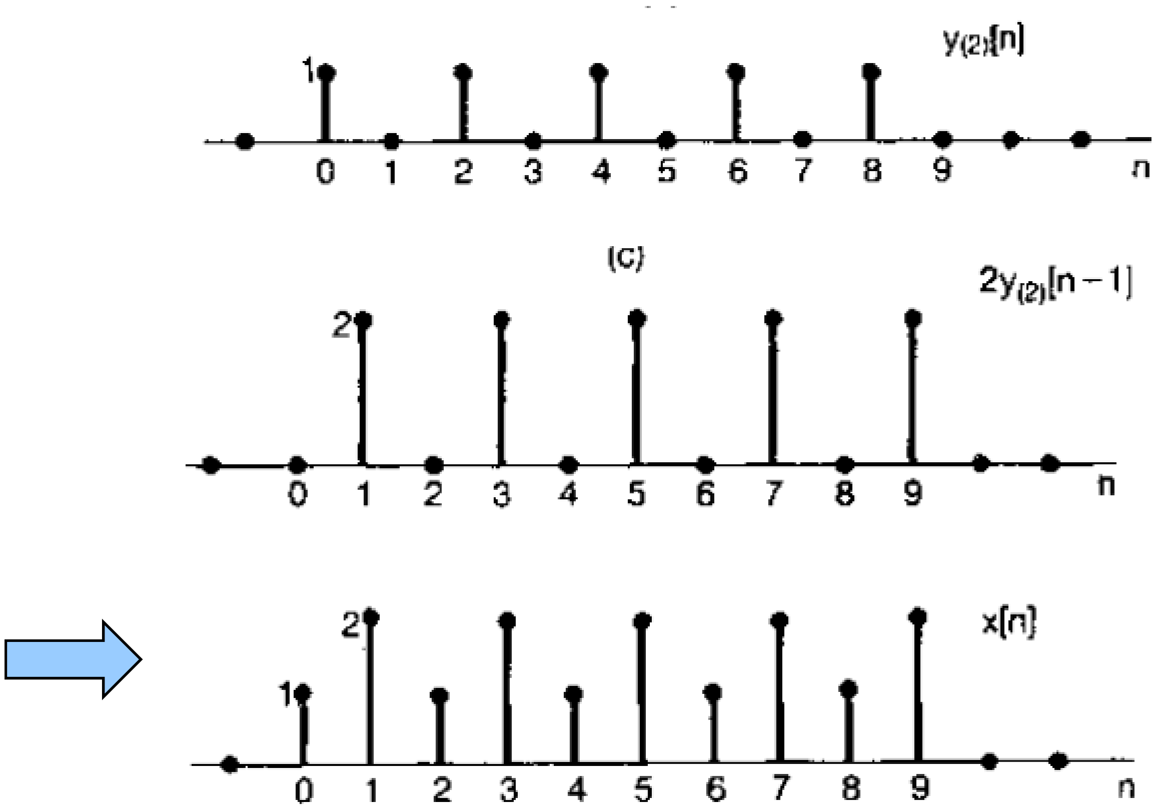
- Example 3.3: we have signal x[n]



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

5.3 Properties

- Example 3.3: we have signal x[n]



5.3 Properties

- Now we consider $y[n] = g[n-2]$ with $N_1 = 2$

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$y_{(2)}[n] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \overset{\mathcal{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$\Rightarrow \quad X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

5.3 Properties

- Differentiation in frequency

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}$$

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

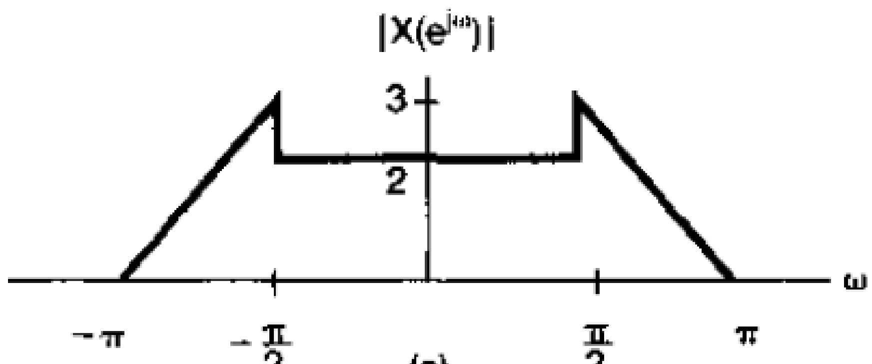
5.3 Properties

- Parselval's relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

5.3 Properties

- Example 3.4: consider the following signal, determine whether or not periodic in time domain, real, even and of finite energy?



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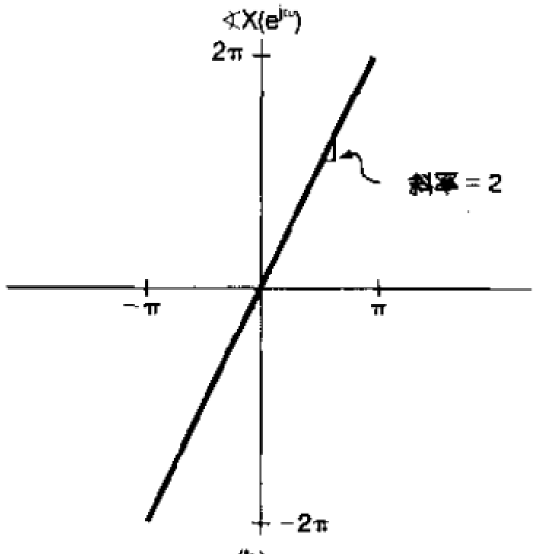


(b)

5.3 Properties

- Not periodic in time domain: since periodicity in time domain implies impulses located at various integer multiples of the fundamental frequency.
- Real in time domain: since the Fourier transform have even magnitude and odd phase function.
- Not even signal: since the following function is not real-valued function.

$$X(j\omega) = |X(e^{j\omega}|e^{-j2\omega}$$

- Finite energy: since the following value is finite

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

5.4 The convolution property

- For discrete-time signals

$$y[n] = x[n] * h[n]$$

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})}$$

  - Map the convolution of two signals to the simple algebraic operation of multiple their Fourier transform
  - Facilitate the analysis of signals and systems
  - The frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency

5.4 The convolution property

- Example 4.1: Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

$$\Rightarrow \quad y[n] = x[n - n_0],$$

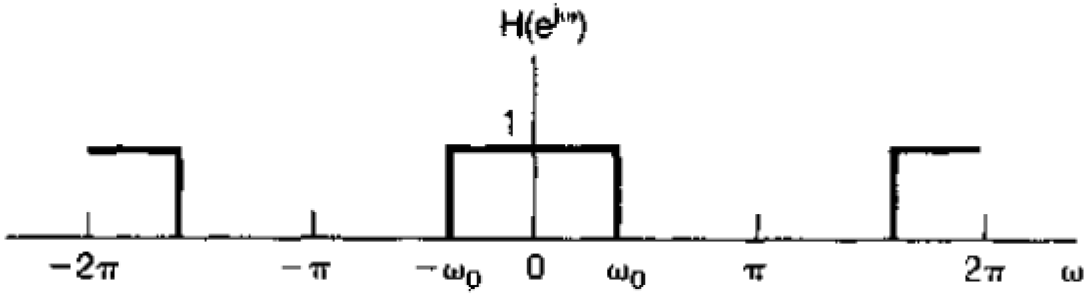
$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

$$\Rightarrow \quad y[n] = x[n - n_0].$$

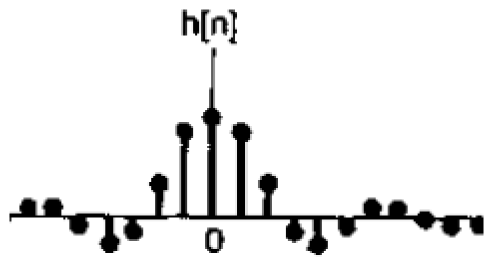
5.4 The convolution property

- Example 4.2: Consider the following frequency response, determine the unit impulse response



(a)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{\sin\omega_c n}{\pi n}$$



5.4 The convolution property

- Example 4.3: Consider the unit impulse response and input are given as follows:

$$h[n] = \alpha^n u[n] \qquad |\alpha| < 1,$$

$$x[n] = \beta^n u[n] \qquad |\beta| < 1,$$

Determine the output.

5.4 The convolution property

- Solution:

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

5.4 The convolution property

- When $\alpha \neq \beta$,

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}} \qquad A = \frac{\alpha}{\alpha-\beta},\ B = -\frac{\beta}{\alpha-\beta}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] = \frac{1}{\alpha-\beta}[\alpha^{n+1}u[n] - \beta^{n+1}u[n]]$$

5.4 The convolution property

- When $\alpha = \beta$

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2$$

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$n\alpha^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{\mathcal{F}}{\longleftrightarrow} je^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$y[n] = (n+1)\alpha^n u[n+1]$$

$$\Rightarrow \quad y[n] = (n+1)\alpha^n u[n]$$

5.5 The multiplication property

- We have

$$x[n]y[n]$$

$$\Downarrow$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

- Above equation corresponds to a periodic convolution.

5.5 The multiplication property

- Example 4.1: consider a signal x[n] which is the product of two signals,

$$x[n] = x_1[n]x_2[n]$$

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

Determine the Fourier transform.

5.5 The multiplication property

- Solution: choosing $-\pi < \theta \leqslant \pi$

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

- Let

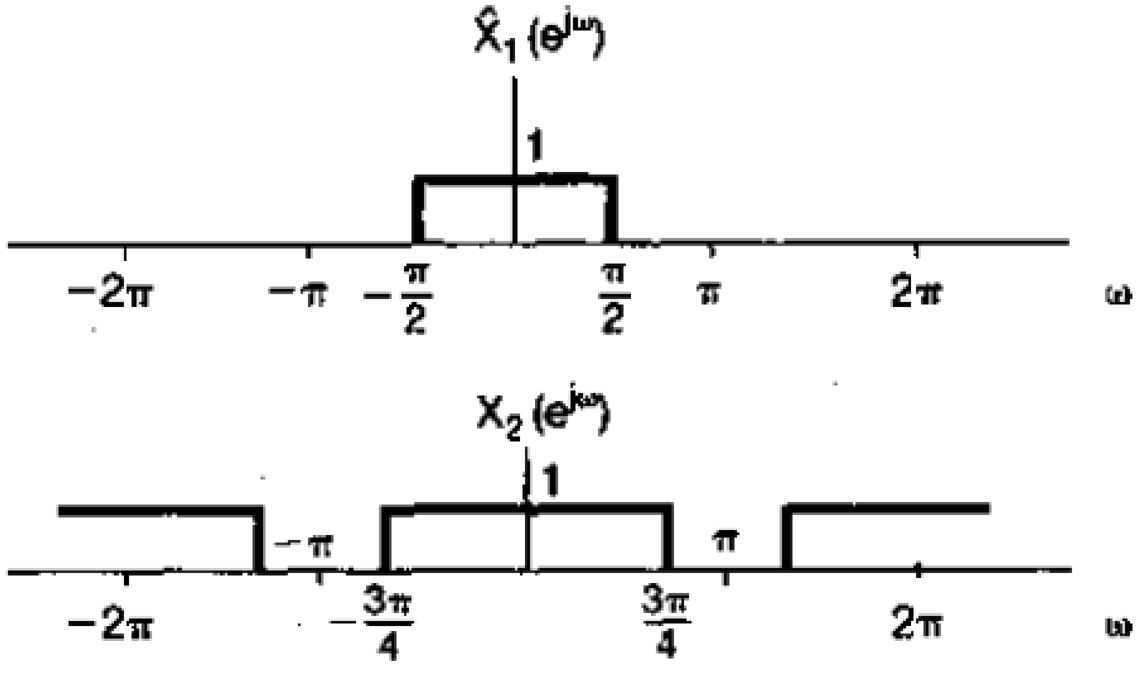
$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), & 对 -\pi < \omega \leqslant \pi \\ 0, & 其余\ \omega \end{cases}$$

- We have

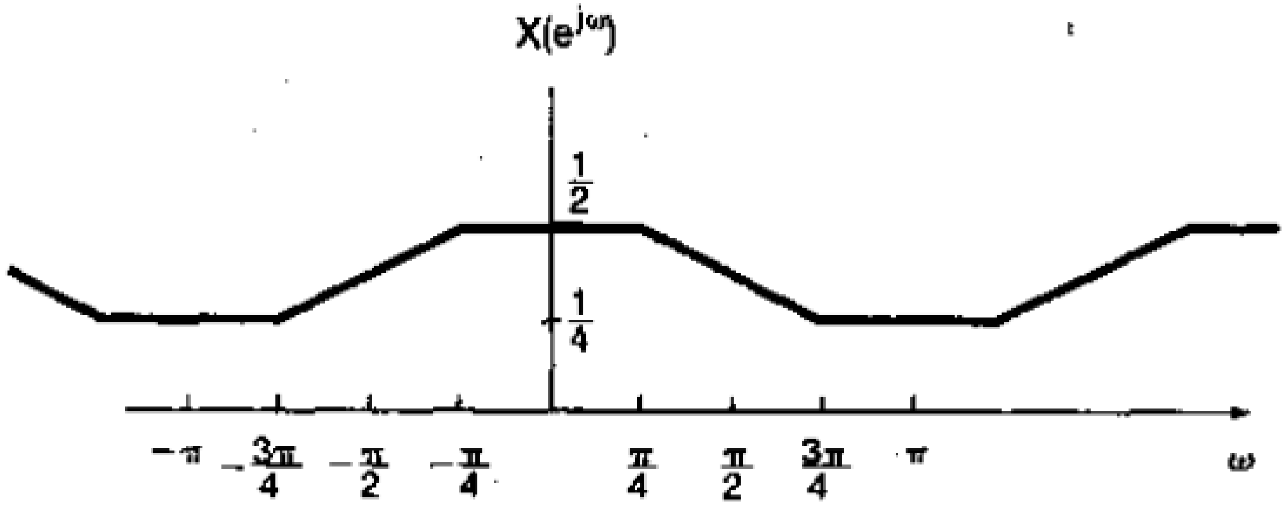
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

5.5 The multiplication property



5.5 The multiplication property



5.6 Summary of the properties

	$x(t)$ → $X(e^{j\omega})$, $y(t)$ → $Y(e^{j\omega})$	
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time reversal	$x[-n]$	$X(e^{-j\omega})$
Time expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{n is multiple of k} \\ 0, & \text{not} \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$

5.6 Summary of the properties

Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in time	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
Differentiation in frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$

5.6 Summary of the properties

Symmetry for Real and Even Signals	x[n] real and even	$X(e^{j\omega})$ real and even
Real and Odd Signals	x[n] real and odd	$X(e^{j\omega})$ imaginary and odd
Even-odd Decomposition of Real Signal [x[n] real]	$x_e[n] = \mathcal{E}v\{x[n]\}$ $x_o[n] = \mathcal{O}d\{x[n]\}$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
Parseval's Relation for nonperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

5.6 Summary of the properties

Signal	Fourier transform	Fourier series coefficients
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \cdots \\ 0, & \text{其余 } k \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ 无理数⇒信号是非周期的
$\cos\omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \cdots \\ 0, & \text{其余 } k \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ 无理数⇒信号是非周期的
$\sin\omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega - \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \cdots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N \cdots \\ 0, & \text{其余 } k \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ 无理数⇒信号是非周期的

5.6 Summary of the properties

Signal	Fourier transform	Fourier series coefficients
$x[n]=1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega-2\pi l)$	$a_k=\begin{cases}1, & k=0,\pm N,\ \pm 2N,\cdots\\ 0, & 其余\ k\end{cases}$
周期方波 $x[n]=\begin{cases}1, & \|n\|\leqslant N_1\\ 0, & N_1<\|n\|\leqslant N/2\end{cases}$ 和 $x[n+N]=x[n]$	$2\pi\sum_{k=-\infty}^{+\infty}a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin[2\pi k/2N]},\ k\neq 0,\pm N,\pm 2N,\cdots$ $a_k=\frac{2N_1+1}{N},\ k=0,\ \pm N,\pm 2N,\cdots$
$\sum_{k=-\infty}^{+\infty}\delta[n-kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$	$a_k=\frac{1}{N}$ 对全部 k

5.6 Summary of the properties

Signal	Fourier transform	Fourier series coefficients
$a^n u[n],\ \lvert a \rvert < 1$	$\dfrac{1}{1 - a e^{-j\omega}}$	—
$x[n] \begin{cases} 1, & \lvert n \rvert \leqslant N_1 \\ 0, & \lvert n \rvert > N_1 \end{cases}$	$\dfrac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\dfrac{\sin Wn}{\pi n} = \dfrac{W}{\pi}\mathrm{sinc}\left(\dfrac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leqslant \lvert \omega \rvert \leqslant W \\ 0, & W < \lvert \omega \rvert \leqslant \pi \end{cases}$ $X(\omega)$周期的，周期为 2π	—
$\delta[n]$	1	—

5.6 Summary of the properties

Signal	Fourier transform	Fourier series coefficients
$u[n]$	$\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{+\infty}\pi\delta(\omega-2\pi k)$	—
$\delta[n-n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n],\ \lvert a\rvert<1$	$\frac{1}{(1-ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!}a^n u[n] < 1$	$\frac{1}{(1-ae^{-j\omega})^r}$	—

5.7 Duality

- Duality in the discrete-time Fourier series
  - The Fourier series coefficients of the periodic sequence a_k are the values of $(1/N)x[-n]$, i.e., are proportional to the values of the original signal reversed in time.

$$f[k] = \frac{1}{N}\sum_{n=\langle N\rangle} g[n]e^{-jk(2\pi/N)n}$$

$$f[n] = \sum_{k=\langle N\rangle} \frac{1}{N}g[-k]e^{jk(2\pi/N)n}$$

5.7 Duality

- Properties

$$x[n-n_0] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k e^{-jk(2\pi/N)n_0}$$

$$e^{jm(2\pi/N)n} x[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} a_{k-m}$$

$$\sum_{r=\langle N\rangle} x[r]y[n-r] \stackrel{\mathcal{FS}}{\longleftrightarrow} N a_k b_k$$

$$x[n]y[n] \stackrel{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l}$$

5.7 Duality

- Example 7.1: determine the Fourier coefficients of following signal with N=9

$$x[n]=\begin{cases}\frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & k\neq 9\text{ 的倍数}\\ \frac{5}{9}, & k=9\text{ 的倍数}\end{cases}$$

- Solution:

$$g[n]=\begin{cases}1, & |n|\leqslant 2\\ 0, & 2<|n|\leqslant 4\end{cases} \Rightarrow b_k=\begin{cases}\frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & n\neq 9\text{ 的倍数}\\ \frac{5}{9}, & n=9\text{ 的倍数}\end{cases}$$

5.7 Duality

- We have

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-j2\pi nk/9}$$

$$x[n] = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-j2\pi nk/9}$$

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2}e^{+j2\pi nk'/9}$$

$$a_k = \begin{cases} 1/9, & |k| \leqslant 2 \\ 0, & 2 < |k| \leqslant 4 \end{cases}$$

5.8 System characterization by Linear constant-coefficient differential equation

- Describe the LTI system with input-output relationship given as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Use Fourier transform to derive the frequency response $H(e^{j\omega})$

5.8 System characterization by Linear constant-coefficient differential equation

- One-way: assume an input $x[n]=e^{j\omega n}$, then the output has a form of $H(e^{j\omega})e^{j\omega n}$.
- We can also use Fourier transform to determine $H(e^{j\omega})$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

5.8 System characterization by Linear constant-coefficient differential equation

- Example 8.1: consider a LTI system characterized by

$$y[n] - ay[n-1] = x[n]$$

Determine the frequency response.

- Solution:

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \quad \Rightarrow \quad h[n] = a^n u[n]$$

5.8 System characterization by Linear constant-coefficient differential equation

- Example 8.2: consider a LTI system characterized by

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$$

Determine the frequency response.

- Solution:

$$H(e^{j\omega})=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}$$

5.8 System characterization by Linear constant-coefficient differential equation

- We factor the denominator as

$$H(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

$$\Rightarrow \quad h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

5.8 System characterization by Linear constant-coefficient differential equation

- Example 8.2: consider a LTI system characterized by

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

Now we have an input of $x[n] = \left(\frac{1}{4}\right)^n u[n]$, determine the output.

5.8 System characterization by Linear constant-coefficient differential equation

- Solution:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

$$\Rightarrow \quad y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$