Fourier Series Representation of Periodic Signals

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Outline

- The response of LIT system to complex exponentials
- Fourier series representation of continuous/discrete-time periodic signals
- Properties of continuous/discrete-time Fourier series
- Filtering

3.1 Introduction

- Representation & analysis of LTI using convolution sum/integral
  - Representing signals as linear combinations of shifted impulses
- In chapters 3, 4, and 5, we learn alternative representation using complex exponentials.
  - It provides us with another convenient way to analyze the system and gain insight into their properties.

3.1 Introduction

- In chapter 3, we focus on
  - Representation of continuous/discrete-time periodic signals
- In chapters 4 and 5, we extend the analysis to
  - Aperiodic signals with finite energy
- These representations provide us
  - More powerful and important tools and insights for analyzing, designing and understanding signals and LTI systems

3.1 The response LTI to complex exponentials

- The importance of complex exponentials in the study of LTI systems stems from the fact:
  - Continuous-time

$$e^{st} \to H(s)e^{st}$$

  - Discrete-time:

$$z^n \to H(z)z^n$$

  - How to prove them?

3.1 The response to LTI complex exponentials

$$x(t) = e^{st}$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$= e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$y(t) = H(s)e^{st}$$

$$x[n] = z^n$$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$= H(z)z^n$$

3.1 The response to LTI complex exponentials

- An example:

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$$

$$a_1 e^{s_1 t} \to a_1 H(s_1) e^{s_1 t}$$

$$a_2 e^{s_2 t} \to a_2 H(s_2) e^{s_2 t}$$

$$a_3 e^{s_3 t} \to a_3 H(s_3) e^{s_3 t}$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

- In general:

$$x(t) = \sum_k a_k e^{s_k t} \qquad x[n] = \sum_k a_k z_k^n$$

$$y(t) = \sum_k a_k H(s_k) e^{s_k t} \qquad y[n] = \sum_k a_k H(z_k) z_k^n$$

3.3 Fourier series representation of continuous-time periodic signals

- 3.3.1 Linear combinations of harmonically related complex exponentials

$$x(t) = x(t + T)$$

- For periodic signal x(t), the minimum positive, non-zero value of T is fundamental period
- $\omega_0 = 2\pi/T$ is referred to as the fundamental frequency
- Harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \qquad k = 0, \pm 1, \pm 2, \cdots$$

3.3 Fourier series representation of continuous-time periodic signals

- 3.3.1 For a signal, the Fourier representation is

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

  - For k = 0, the term is a constant
  - For K=1 or -1, the terms is called the first harmonic component
  - For K=2 or -2, the terms is called the second harmonic component
  - For K=N or -N, the terms is called the N-th harmonic component

3.3 Fourier series representation of continuous-time periodic signals

- For real signals, we have

$$a_k^* = a_{-k}$$

- And with $a_k = A_k e^{j\theta_k}$, we have

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathcal{Re}\{A_k e^{j(k\omega_0 t + \theta_k)}\}$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

3.3 Fourier series representation of continuous-time periodic signals

$$x^*(t) = x(t)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t}$$

3.3 Fourier series representation of continuous-time periodic signals

- 3.3.2 Determine the Fourier Series Representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t}$$

  - Then we have

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

  - Referred to as Fourier series coefficients

$$a_0 = \frac{1}{T}\int_T x(t)dt$$

  - Is the constant component of x(t)

3.3 Fourier series representation of continuous-time periodic signals

- Proof:

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t\,dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

3.3 Fourier series representation of continuous-time periodic signals

- Example 1: consider a signal,

$$x(t) = \sin\omega_0 t$$

Determine the Fourier series coefficient.

3.3 Fourier series representation of continuous-time periodic signals

- Solution:

$$\sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

We have

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}$$

$$a_k = 0, \quad k \neq +1 \text{ 或 } -1$$

3.3 Fourier series representation of continuous-time periodic signals

- Example 2: consider a signal,

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

Determine the Fourier series coefficient.

3.3 Fourier series representation of continuous-time periodic signals

- Solution:

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right]$$

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}$$

We have

$$a_0 = 1,\ a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,\ a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$$

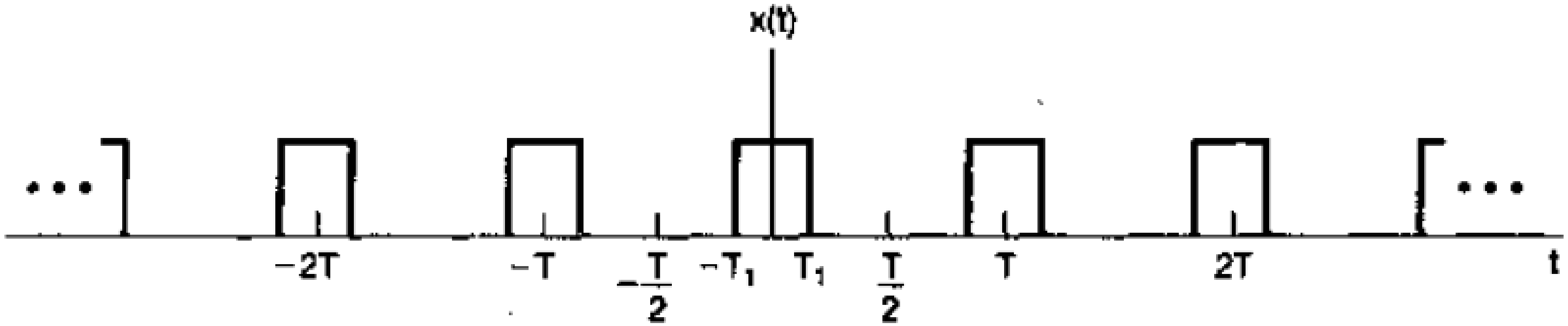
$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),\ a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),\ a_k = 0,\ |k| > 2$$

3.3 Fourier series representation of continuous-time periodic signals

- Example 3: consider a periodic square wave,

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Determine the Fourier series coefficient.



3.3 Fourier series representation of continuous-time periodic signals

- Solution:

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} \mathrm{d}t = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} \mathrm{e}^{-\mathrm{j}k\omega_0 t}\mathrm{d}t = -\frac{1}{\mathrm{j}k\omega_0 T}\mathrm{e}^{-\mathrm{j}k\omega_0 t}\Big|_{-T_1}^{T_1}$$

$$= \frac{2}{k\omega_0 T}\left[\frac{\mathrm{e}^{\mathrm{j}k\omega_0 T_1} - \mathrm{e}^{-\mathrm{j}k\omega_0 T_1}}{2\mathrm{j}}\right]$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

3.4 Convergence of the Fourier representation

- In some cases, the integral in obtaining the coefficient is not convergent.
- Fortunately, there are no convergence difficulties for large classes of periodical signals.
- In most of cases, the periodic signals that can be represented by Fourier series is the signal with finite energy over a single period

$$\int_T |x(t)|^2 dt < \infty$$

3.4 Convergence of the Fourier representation

- Dirichlet conditions:
  - Condition1 : Over any period, x(t) must be absolutely integrable, that is

$$\int_T |x(t)|\,\mathrm{d}t < \infty$$

Which makes sure

$$|a_k| \leqslant \frac{1}{T}\int_T |x(t)e^{jk\omega_0 t}|\,\mathrm{d}t = \frac{1}{T}\int_T |x(t)|\,\mathrm{d}t$$

An example:

$$x(t) = \frac{1}{t}, \quad 0 < t \leqslant 1$$

3.4 Convergence of the Fourier representation

- Dirichlet conditions:
  - Condition 2 : In any finite interval of time, x(t) is of bounded variation, that is there are no more than a finite number of maxima and minima during any single period of the signal.

An example:

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leqslant 1$$

3.4 Convergence of the Fourier representation

- Dirichlet conditions:
  - Condition 3 : In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

- In practice, more of signals are convergent. For this reason, the question of convergence of Fourier series will not play a significant role in the remainder of the book.

3.5 Properties of continuous-time Fourier Series

- Assumption: Fundamental period T, fundamental frequency is $\omega_0 = 2\pi / T$

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

denotes a periodic signal and its Fourier series coefficient.

- Condition 1: Linearity (with the same period T)

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xrightarrow{\mathcal{FS}} b_k$$

$$\Rightarrow \quad z(t) = Ax(t) + By(t) \xrightarrow{\mathcal{FS}} C_k = Aa_k + Bb_k$$

3.5 Properties of continuous-time Fourier Series

- Condition 2: Time shifting

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

$$\Rightarrow \quad x(t-t_0) \xrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

Proof:

$$b_k = \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt$$

$$\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau = e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

3.5 Properties of continuous-time Fourier Series

- Condition 3: Time reversal

$$x(t) \xrightarrow{\mathscr{FS}} a_k$$

$$\Rightarrow \quad x(-t) \xrightarrow{\mathscr{FS}} a_{-k}$$

Proof:

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}$$

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}$$

3.5 Properties of continuous-time Fourier Series

- For even signal, its Fourier series coefficients are also even, i.e.,

$$a_{-k} = a_k$$

- For odd signal, its Fourier series coefficients are also odd, i.e.,

$$a_{-k} = -a_k$$

3.5 Properties of continuous-time Fourier Series

- Condition 4: Time scaling

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Rightarrow \quad x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

Proof: the fundamental period and the fundamental frequency are changed.

3.5 Properties of continuous-time Fourier Series

- Condition 5: multiplication

$$x(t) \xrightarrow{\mathcal{FS}} a_k$$

$$y(t) \xrightarrow{\mathcal{FS}} b_k$$

$$\Rightarrow \quad x(t)y(t) \xrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Proof:

3.5 Properties of continuous-time Fourier Series

- Condition 6: conjugation & conjugate symmetry

$$x(t) \xrightarrow{\mathcal{F}} a_k$$

$$\Rightarrow \quad x^*(t) \xrightarrow{\mathcal{FS}} a^*_{-k}$$

- With this property, we have if $x(t) = x^*(t)$ (real signal)

$$a_{-k} = a_k^*$$

- If x(t) is real and even, we have real

$$a_k^* = a_{-k} \qquad a_k = a_k^*$$

3.5 Properties of continuous-time Fourier Series

- Condition 7: Parseval's relation for continuous-time periodic signal

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

- Also

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

3.5 Properties of continuous-time Fourier Series

- Summary:

X(t) and y(t) are periodic and with period T & $\omega_0 = 2\pi/T$		
Linearity	$Ax(t)+By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t-t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency shifting	$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time reversal	$x(-t)$	a_{-k}
Time Scaling (Period T/a)	$x(at), a>0$	a_k
Periodic Convolution	$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$

3.5 Properties of continuous-time Fourier Series

X(t) and y(t) are periodic and with period T & $\omega_0 = 2\pi/T$		
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration (finite and periodic only if $a_0 = 0$	$\int_{-\infty}^{t} x(t)dt$	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$

3.5 Properties of continuous-time Fourier Series

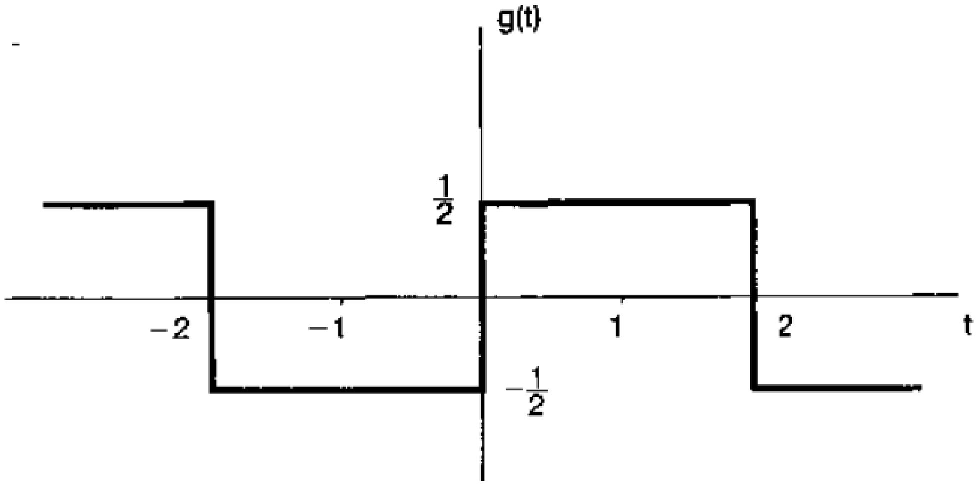
X(t) and y(t) are periodic and with period T & $\omega_0 = 2\pi/T$		
Real and Even Signals	X(t) real and even	a_k real and even
Real and Odd Signals	X(t) real and odd	a_k imaginary and odd
Even-odd Decomposition of Real Signal [x(t) real]	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} \\ x_0(t) = \mathcal{O}d\{x(t)\} \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$
Parseval's Relation for Periodic Signals	$\frac{1}{T}\int_T \mid x(t)\mid^2 dt = \sum_{k=-\infty}^{+\infty} \mid a_k \mid^2$	

3.5 Properties of continuous-time Fourier Series

- Example 1: consider a signal,

$$g(t) = x(t-1) - \frac{1}{2}$$

Determine the Fourier series coefficient.



3.5 Properties of continuous-time Fourier Series

- Solution: with time-shift property, the Fourier coefficient x(t-1) can be expressed as

$$b_k = a_k e^{-jk\pi/2}$$

For the constant offset 1/2

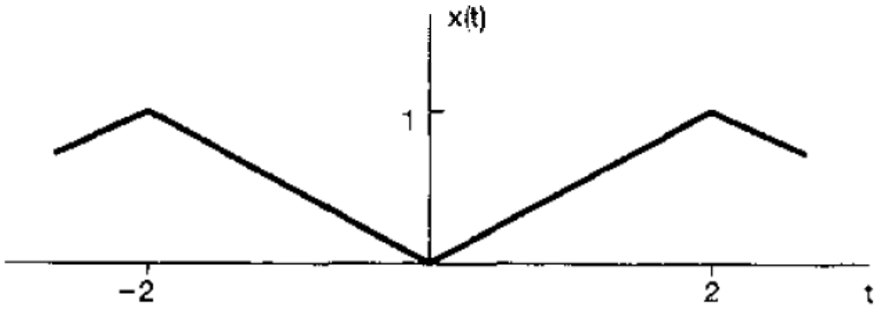
$$c_k = \begin{cases} 0, & k \neq 0 \\ -\frac{1}{2}, & k = 0 \end{cases}$$

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & k \neq 0 \\ a_0 = \frac{1}{2}, & k = 0 \end{cases}$$

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & k \neq 0 \\ 0, & k = 0 \end{cases}$$

3.5 Properties of continuous-time Fourier Series

- Example 2: consider a signal,



Determine the Fourier series coefficient.

3.5 Properties of continuous-time Fourier Series

- Solution:

$$d_k = jk(\pi/2)e_k$$

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0$$

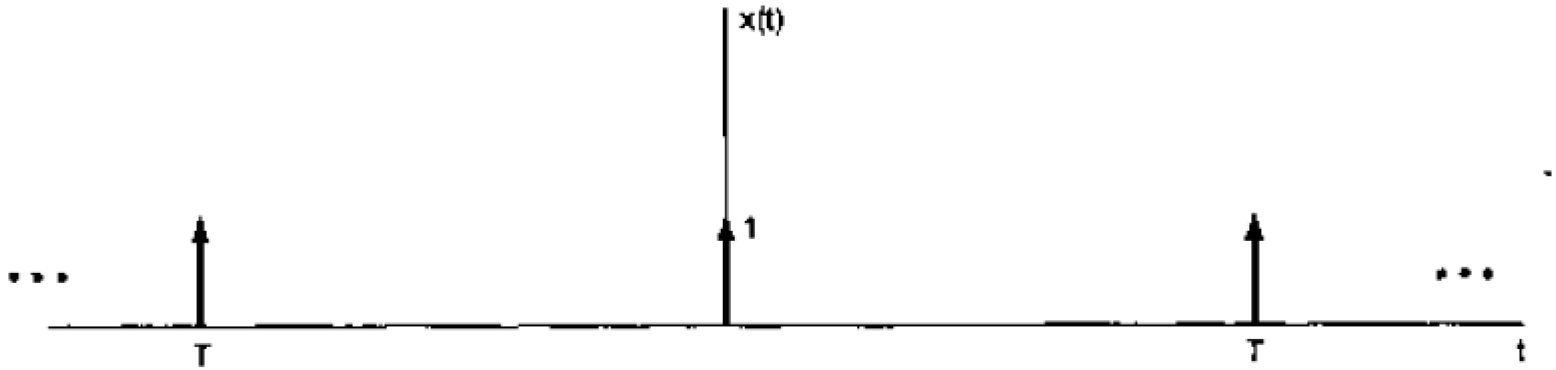
$$e_0 = \frac{1}{2}$$

3.5 Properties of continuous-time Fourier Series

- Example 3: consider a signal,

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

Determine the Fourier series coefficient.



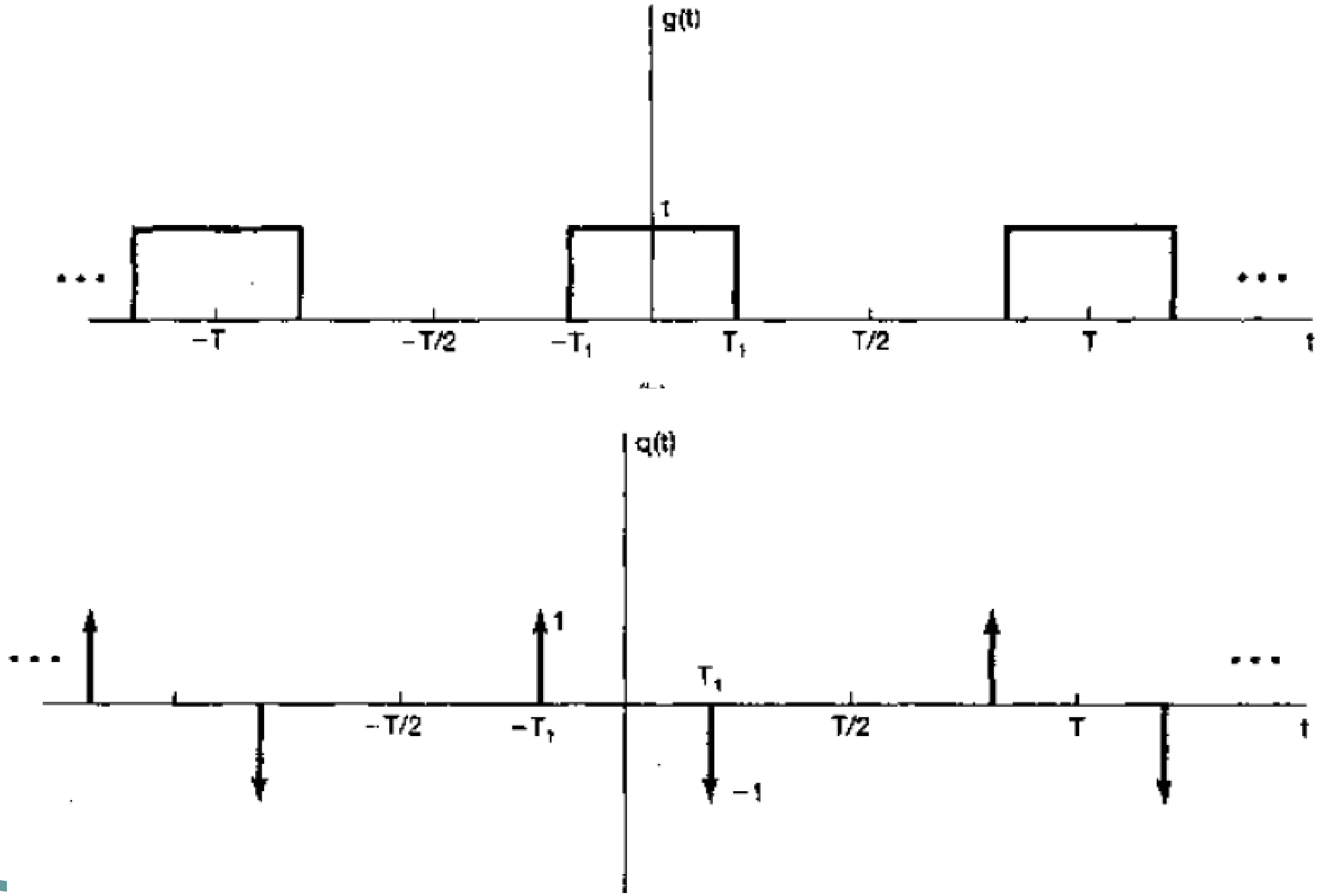
3.5 Properties of continuous-time Fourier Series

- Solution1:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\pi t/T} dt = \frac{1}{T}$$

3.5 Properties of continuous-time Fourier Series

- Solution 2:



3.5 Properties of continuous-time Fourier Series

- Solution 2:

$$q(t) = x(t + T_1) - x(t - T_1)$$

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k$$

$$b_k = \frac{1}{T}[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}] = \frac{2j\sin(k\omega_0 T_1)}{T}$$

$$b_k = jk\omega_0 c_k$$

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

$$c_0 = \frac{2T_1}{T}$$

3.6 Fourier series representation of discrete-time periodic signals

- The Fourier series representation of a discrete-time periodic is a finite series.
- All of the following complex exponential have fundamental frequencies that are multiples of $2\pi/N$

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \cdots$$

- We have

$$\phi_k[n] = \phi_{k+rN}[n]$$

3.6 Fourier series representation of discrete-time periodic signals

- The Fourier series representation of discrete-time signals

$$x[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

- The problem to obtain a_k

$$x[0] = \sum_{K=\langle N\rangle} a_k$$

$$x[1] = \sum_{K=\langle N\rangle} a_k e^{j2\pi k/N}$$

$$\vdots$$

$$x[N-1] = \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}$$

3.6 Fourier series representation of discrete-time periodic signals

- We have a closed-form expression for obtaining discrete-time Fourier series pair

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n}$$

3.6 Fourier series representation of discrete-time periodic signals

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \cdots, \\ 0, & \text{其余 } k \end{cases}$$

3.6 Fourier series representation of discrete-time periodic signals

- Example 1: consider a signal,

$$x[n] = \sin \omega_0 n$$

Determine the Fourier series coefficient.

- Solution: when $2\pi/\omega_0$ is an integer,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}$$

3.6 Fourier series representation of discrete-time periodic signals

- When $2\pi/\omega_0$ is a ratio of integers

$$\omega_0 = \frac{2\pi M}{N}$$

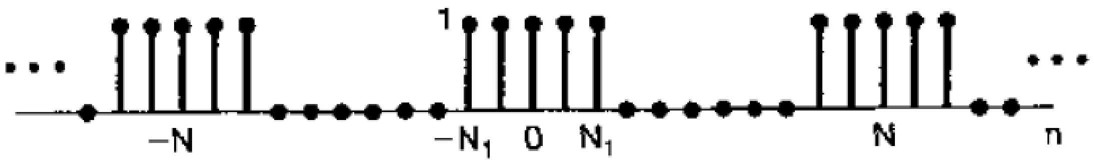
Then we have

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n}$$

$$a_M = (1/2j),\ a_{-M} = (-1/2j),$$

3.6 Fourier series representation of discrete-time periodic signals

- Example 2: consider a discrete-time periodic square wave



Determine the Fourier series coefficient.

3.6 Fourier series representation of discrete-time periodic signals

- Solution:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

Letting m = n+N_1

$$a_k = \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N} e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

3.6 Fourier series representation of discrete-time periodic signals

- And

$$a_k = \frac{1}{N} e^{jk(2\pi/N)N_1} \left(\frac{1 - e^{-jk2\pi(2N_1+1)/N}}{1 - e^{-jk(2\pi/N)}} \right)$$

$$= \frac{1}{N} \frac{e^{-jk(2\pi/2N)}\left[e^{jk2\pi(N_1+1/2)/N} - e^{-jk2\pi(N_1+1/2)/N}\right]}{e^{-jk(2\pi/2N)}\left[e^{jk(2\pi/2N)} - e^{-jk(2\pi/2N)}\right]}$$

$$= \frac{1}{N} \frac{\sin[2\pi k(N_1 + 1/2)/N]}{\sin(\pi k/N)}, \quad k \neq 0, \pm N, \pm 2N, \cdots$$

and

$$a_k = \frac{2N_1 + 1}{N}, \quad k = 0, \pm N, \pm 2N, \cdots$$

3.7 Properties of discrete-time Fourier series

- Summary

X[n] and y[n] are periodic and with period N & $\omega_0 = 2\pi/N$		
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases}$	$\frac{a_k}{m}$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r] y[n-r]$	$Na_k b_k$

3.5 Properties of continuous-time Fourier Series

X[n] and y[n] are periodic and with period N & $\omega_0 = 2\pi/N$		
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First difference	$x[n]-x[n-1]$	$(1-e^{-jk(2\pi/N)})a_k$
Running sum (finite and periodic only if $a_0=0$	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a_{-k}^* \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$

3.5 Properties of continuous-time Fourier Series

X(t) and y(t) are periodic and with period T & $\omega_0 = 2\pi/T$		
Real and Even Signals	X[n] real and even	a_k real and even
Real and Odd Signals	X[n] real and odd	a_k imaginary and odd
Even-odd Decomposition of Real Signal [x[n] real]	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} [\\ x_0(t) = \mathcal{O}d\{x(t)\} \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$
Parseval's Relation for Periodic Signals	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$	

3.7 Properties of discrete-time Fourier series

- Multiplication
  - Assume

$$x[n] \xrightarrow{\mathcal{FS}} a_k$$

$$y[n] \xrightarrow{\mathcal{FS}} b_k$$

  - We have

$$x[n]y[n] \xrightarrow{\mathcal{FS}} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

3.7 Properties of discrete-time Fourier series

- First difference
  - Assume

$$x[n] \xleftrightarrow{\mathcal{FS}} a_k$$

  - We have

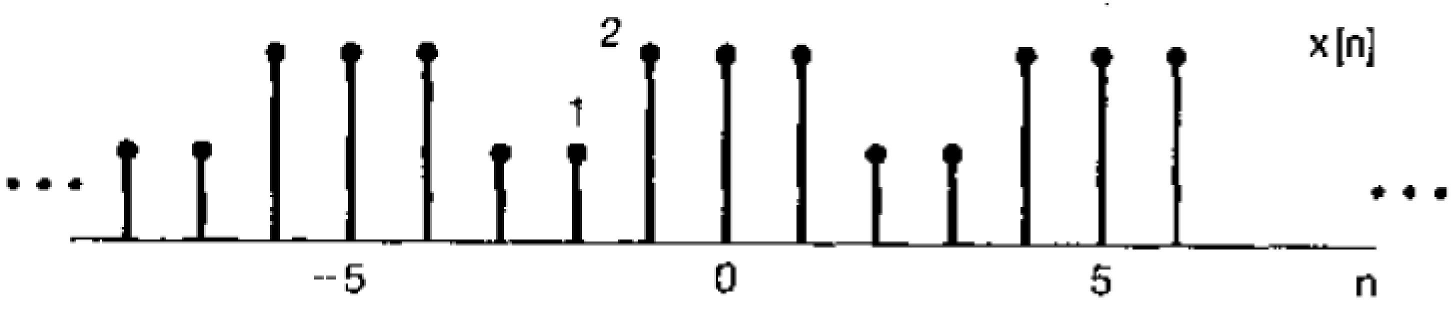
$$x[n] - x[n-1] \xleftrightarrow{\mathcal{FS}} (1 - e^{-jk(2\pi/N)})a_k$$

3.7 Properties of discrete-time Fourier series

- Parseval's relation
  - $$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

3.7 Properties of discrete-time Fourier series

- Example 1:
  - consider a signal

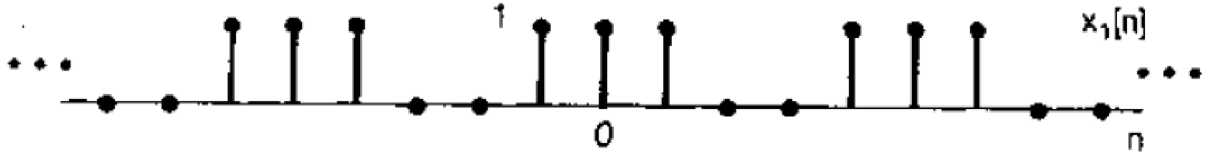


(a)

  - Determine the Fourier series coefficient.

3.7 Properties of discrete-time Fourier series

- solution:



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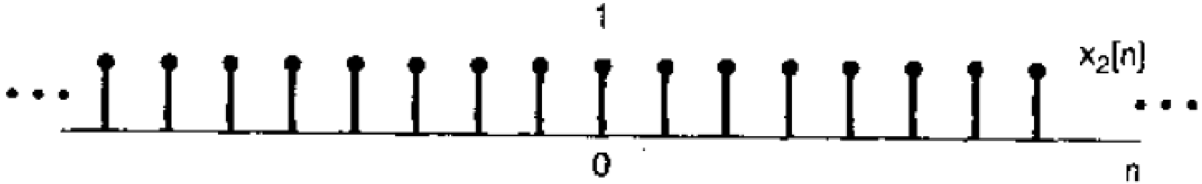


(c)

$$a_k = b_k + c_k$$

3.7 Properties of discrete-time Fourier series

- solution:
  -

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & k \neq 0, \pm 5, \pm 10, \cdots \\ \dfrac{3}{5}, & k = 0, \pm 5, \pm 10, \cdots \end{cases}$$

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1$$

$$a_k = \begin{cases} b_k = \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & k \neq 0, \pm 5, \pm 10, \cdots \\ \dfrac{8}{5}, & k = 0, \pm 5, \pm 10, \cdots \end{cases}$$

3.7 Properties of discrete-time Fourier series

- Example 2:
  - Consider a signal x[n] with
    - X[n] is periodic with period N = 6

$$\sum_{n=0}^{5} x[n] = 2$$

$$\sum_{n=2}^{7} (-1)^n x[n] = 1$$

    - X[n] has the minimum power per period among the set of signals satisfying the preceding three conditions

3.7 Properties of discrete-time Fourier series

- solution:
  - From condition 2, we conclude

$$a_0 = 1/3,$$

  - From condition 3, we have

$$(-1)^n = e^{-j\pi n} = e^{-j(2\pi/6)3n}$$

so $$a_3 = 1/6,$$

  - Since the power is $P = \sum_{k=0}^{5} |a_k|^2$, to minimize the power

$$a_1 = a_2 = a_4 = a_5 = 0$$

  - We have

$$x[n] = a_0 + a_3 e^{j\pi n}$$
$$= (1/3) + (1/6)(-1)^n$$

3.8 Fourier series and LTI systems

- For continuous-time case, with input $x(t)=e^{st}$;

$$y(t)=H(s)e^{st},$$

where

$$H(s)=\int_{-\infty}^{+\infty}h(\tau)e^{-s\tau}d\tau$$

- For discrete-time case, with input $x[n]=z^n$

$$y[n]=H(z)z^n$$

where

$$H(z)=\sum_{k=-\infty}^{+\infty}h[k]z^{-k}$$

- We call H(s) and H(z) as system function

3.8 Fourier series and LTI systems

- We focus on $s = j\omega$ and $z = e^{j\omega}$

$$e^{st} = e^{j\omega t}$$

$$z^n = e^{j\omega n}$$

are complex exponential signals at frequency ω

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt \qquad H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}$$

- We call $H(j\omega)$ and $H(e^{j\omega})$ as system function

3.8 Fourier series and LTI systems

- Based on that, with input

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

We have

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

- [1] y(t) has the same fundamental frequency as x(t)
- [2] if $\{a_k\}$ is the set of Fourier series coefficient, then $\{a_k H(jk\omega_0)\}$ is the set of Fourier series coefficients for the output.

3.8 Fourier series and LTI systems

- Based on that, with input

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

We have

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{j2(\pi k/N)})e^{jk(2\pi/N)n}$$

3.8 Fourier series and LTI systems

- Example 1: with input

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

$$a_0 = 1, \quad a_1 = a_{-1} = \frac{1}{4}, \quad a_2 = a_{-2} = \frac{1}{2}, \quad a_3 = a_{-3} = \frac{1}{3}$$

and system unit impulse response being

$$h(t) = e^{-t}u(t)$$

Determine the Fourier series coefficients of output.

3.8 Fourier series and LTI systems

- Solution: we first compute the frequency response

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau = -\frac{1}{1+j\omega} e^{-\tau} e^{-j\omega\tau} \Big|_0^{\infty} = \frac{1}{1+j\omega}$$

The output is given by

$$y(t) = \sum_{k=-3}^{+3} b_k e^{jk2\pi t}$$

$$b_k = a_k H(jk2\pi)$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \qquad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right)$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right)$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

3.8 Fourier series and LTI systems

- Example 2: with input

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

and system unit impulse response being

$$h[n] = \alpha^n u[n], -1 < \alpha < 1,$$

Determine the Fourier series coefficients of output.

3.8 Fourier series and LTI systems

- Solution: we first compute the Fourier series representation

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}$$

and the frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

3.8 Fourier series and LTI systems

The output is given by

$$y[n] = \frac{1}{2}H(e^{j2\pi/N})e^{j(2\pi/N)n} + \frac{1}{2}H(e^{-j2\pi/N})e^{-j(2\pi/N)n}$$

$$= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}$$

3.9 Filtering

- Filtering: change the relative amplitude of the frequency component in a signal or eliminate some frequency components
  - Frequency-shaping filter: a LTI system which changes the shape of input spectrum
  - Frequency-selective filter: pass some frequencies undistorted and significantly attenuate or eliminate others
- Why we can do this?

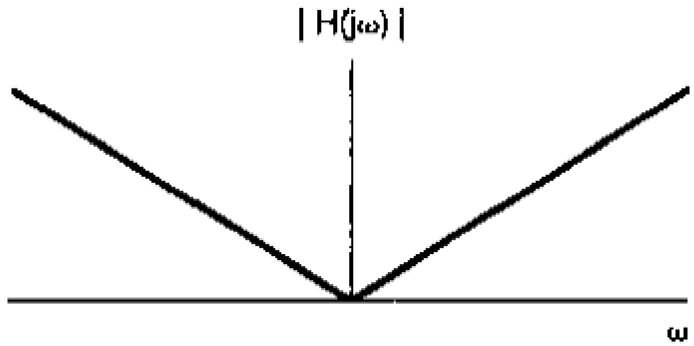
$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

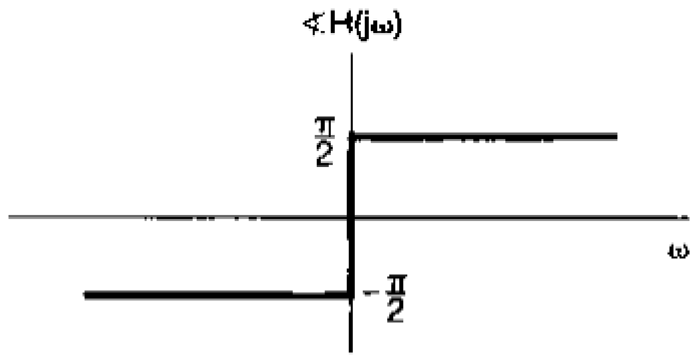
3.9 Filtering

- Frequency-shaping filters
  - An example: Differentiating filter

$$y(t) = dx(t)/dt$$

$$H(j\omega) = j\omega$$





3.9 Filtering

- Differentiating filter can
  - A complex exponential input $e^{j\omega t}$ will receive a greater amplification for large values of ω
  - Then, this filter will enhance the rapid variations in a signal
  - Often used to enhance the edges in image processing

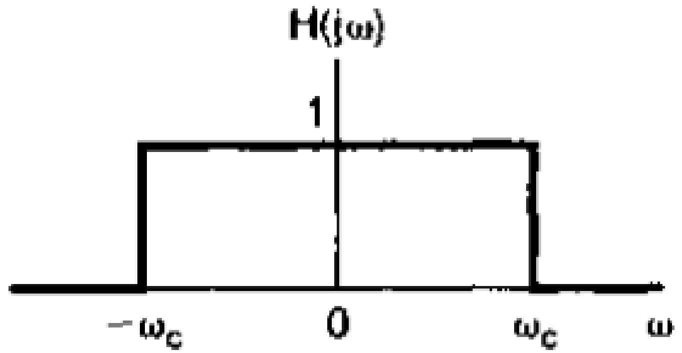
3.9 Filtering

3.9 Filtering

- Frequency-selective filters: pass some frequencies undistorted and significantly attenuate or eliminate others
  - An example: reduce the noise in music or voice recording system
  - Lowpass filter: pass low frequencies and attenuate or reject high frequencies
  - Highpass filter: pass high frequencies and attenuate or reject low frequencies

3.9 Filtering

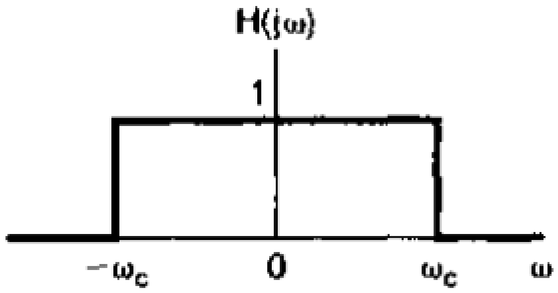
- Cutoff frequency: the frequency in the boundaries between frequencies that are passed and the frequencies that are rejected
- passband & stopband



stopband passband stopband

3.9 Filtering

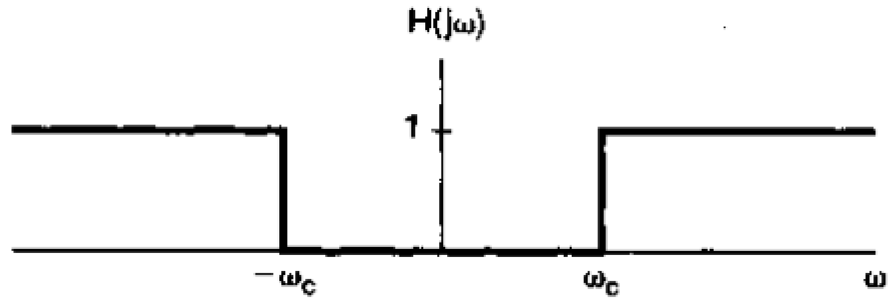
- For continuous-time case:
  - Idea lowpass filter:



$$H(j\omega) = \begin{cases} 1, & |\omega| \leqslant \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

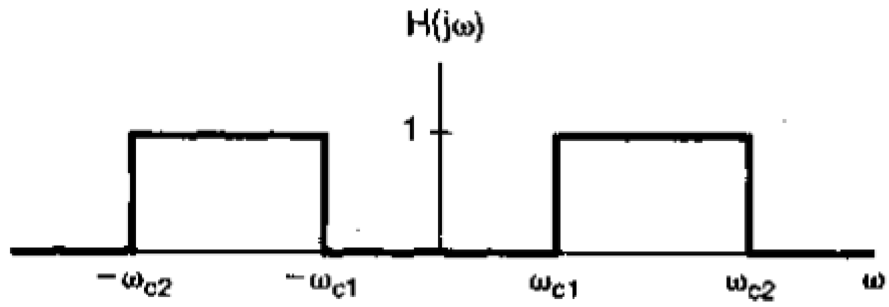
3.9 Filtering

- For continuous-time case:
  - Idea highpass filter with cutoff frequency ω_c



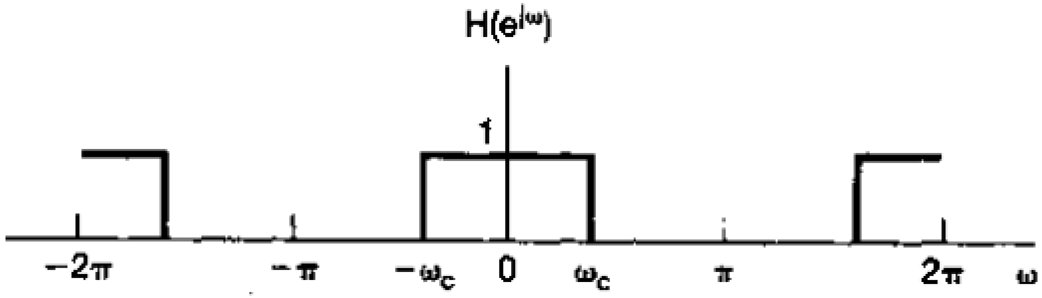
(a)

  - Idea bandpass filter



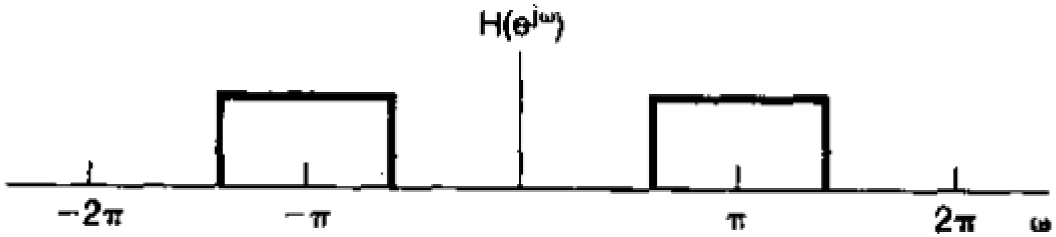
3.9 Filtering

- For discrete-time case:
  - Idea lowpass filter:

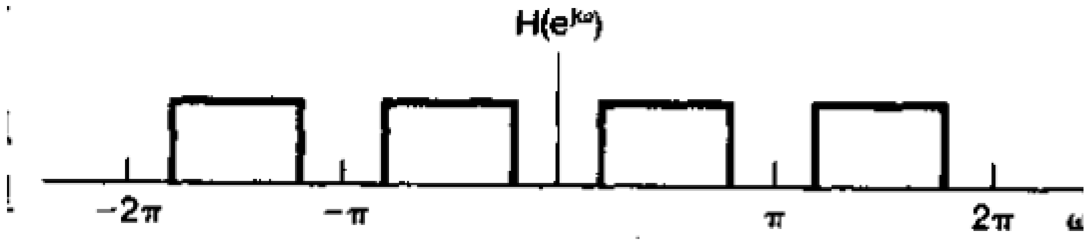


3.9 Filtering

- For discrete-time case:
  - Idea highpass filter with cutoff frequency ω_c



  - Idea bandpass filter

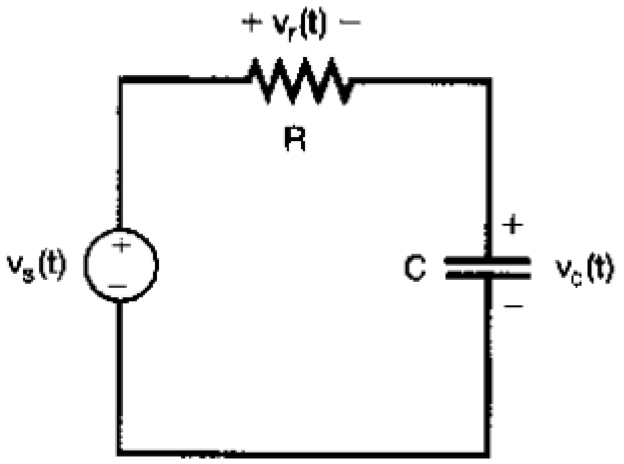


3.10 Examples of continuous-time filters described by differential equations

- In many applications, frequency-selective filtering is accomplished by use of LTI systems described by
  - Linear constant-coefficient differential equations
  - Linear constant-coefficient difference equations

3.10 Examples of continuous-time filters described by differential equations

- A simple RC lowpass filter
  - First-order RC circuit



  - The output voltage is related to the input voltage through the following linear constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

3.10 Examples of continuous-time filters described by differential equations

- Determine the frequency response $H(j\omega)$

$$RC\frac{d}{dt}[H(j\omega)e^{j\omega t}] + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

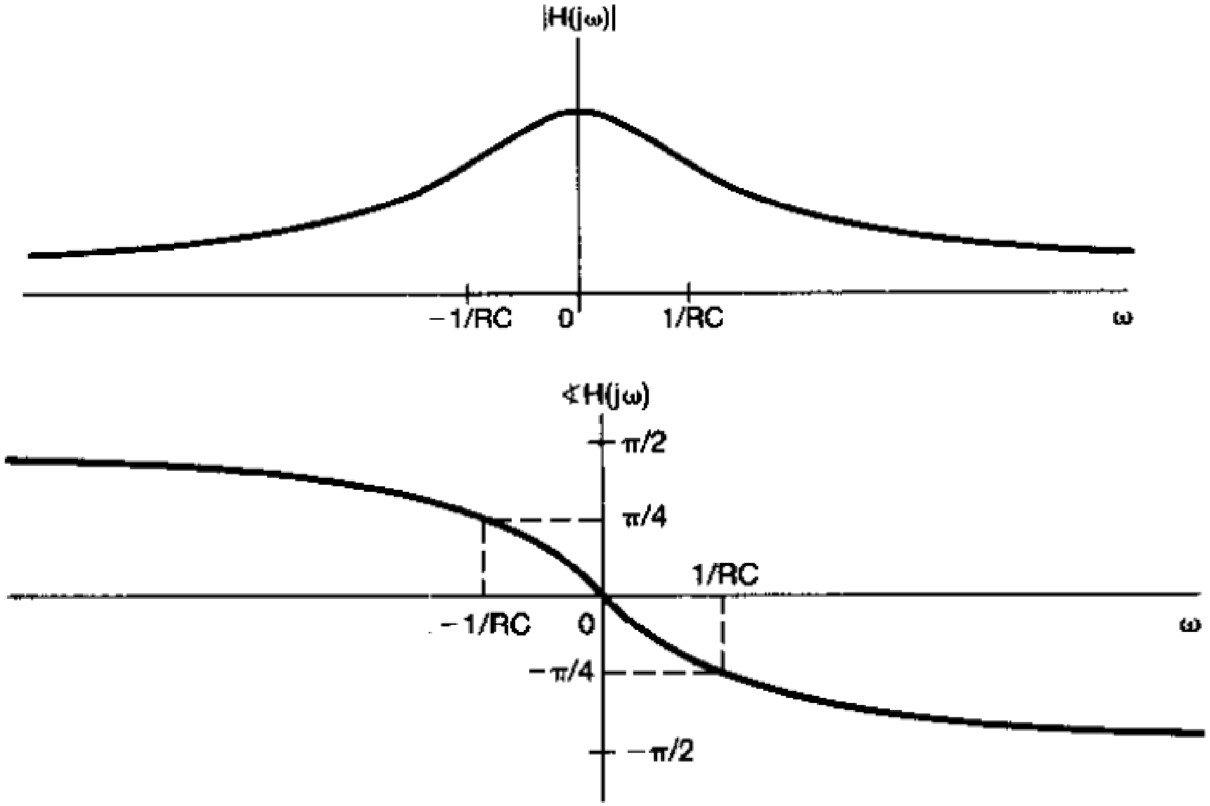
$$\Rightarrow \quad RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$

$$\Rightarrow \quad H(j\omega)e^{j\omega t} = \frac{1}{1 + RCj\omega}e^{j\omega t}$$

$$\Rightarrow \quad H(j\omega) = \frac{1}{1 + RCj\omega}$$

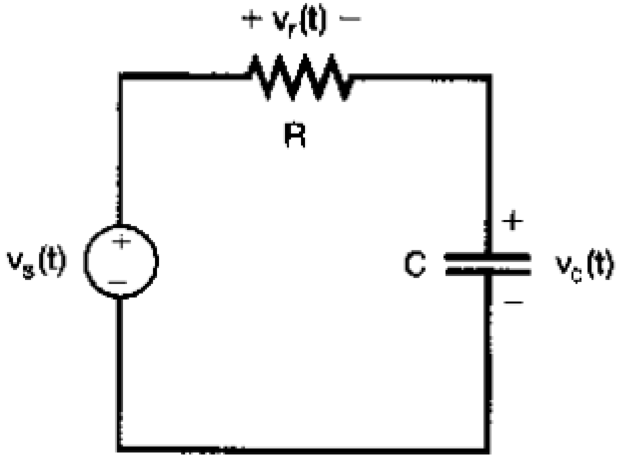
3.10 Examples of continuous-time filters described by differential equations

- The magnitude and phase of frequency response $H(j\omega)$ are



3.10 Examples of continuous-time filters described by differential equations

- A simple RC highpass filter
  - First-order RC circuit



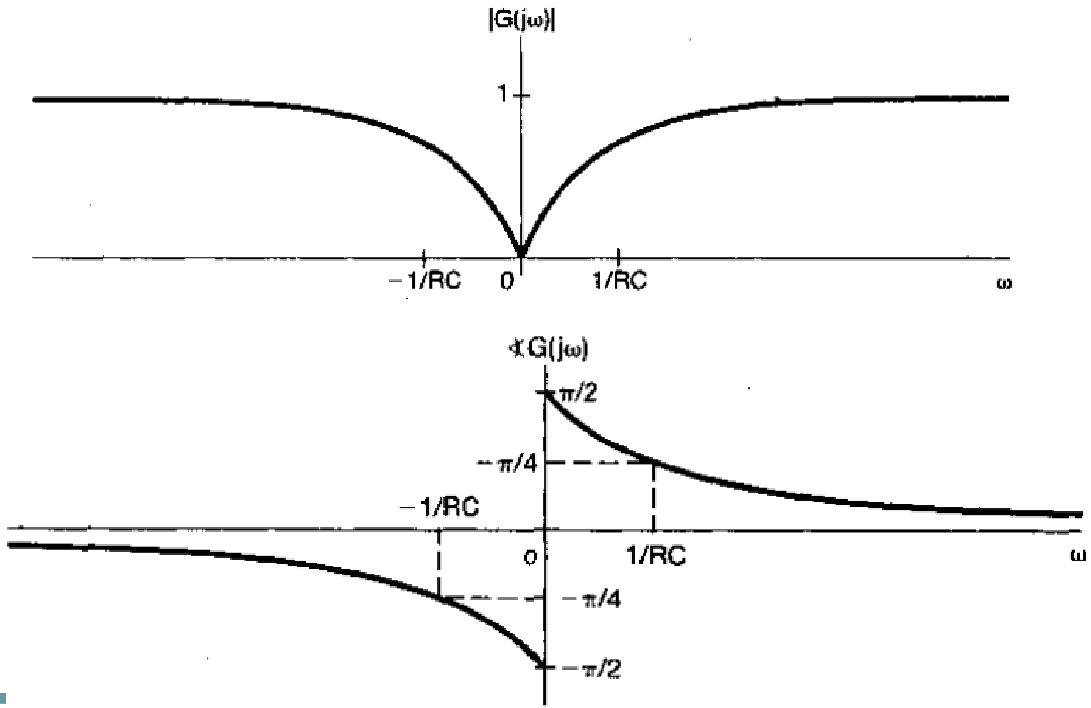
  - The output voltage is related to the input voltage through the following linear constant-coefficient differential equation

$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}$$

3.10 Examples of continuous-time filters described by differential equations

- Determine the frequency response $H(j\omega)$

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$



3.11 Examples of discrete-time filters described by differential equations

- The discrete-time system described by difference equations can be
  - Recursive and have impulse responses of infinite system (IIR systems)
  - Nonrecursive have finite-length impulse responses (FIR systems)

3.11 Examples of discrete-time filters described by differential equations

- First-order recursive discrete-time filters

$$y[n] - ay[n-1] = x[n]$$

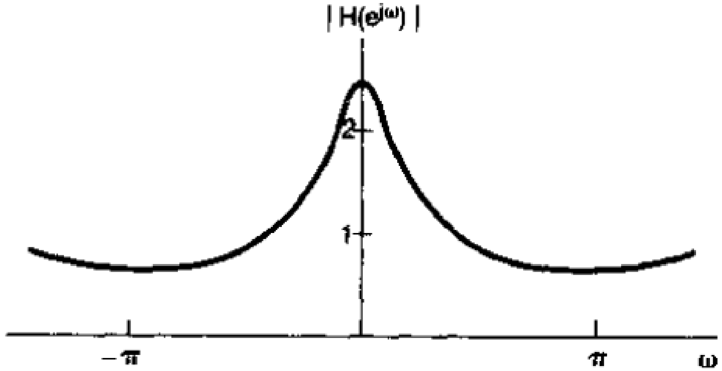
$$\Rightarrow \quad H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

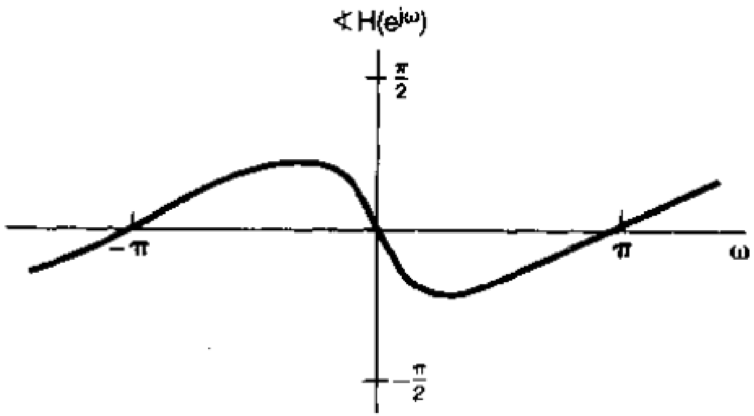
$$\Rightarrow \quad [1 - ae^{-j\omega}]H(e^{j\omega})e^{j\omega n} = e^{j\omega n}$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

3.10 Examples of continuous-time filters described by differential equations

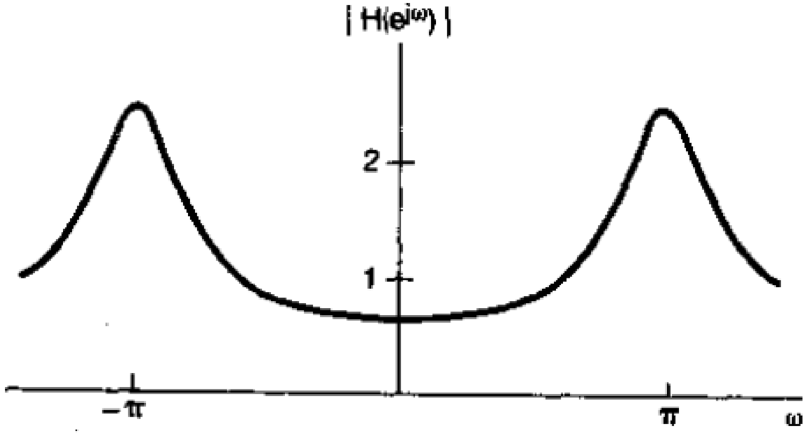
- When $a = 0.6$

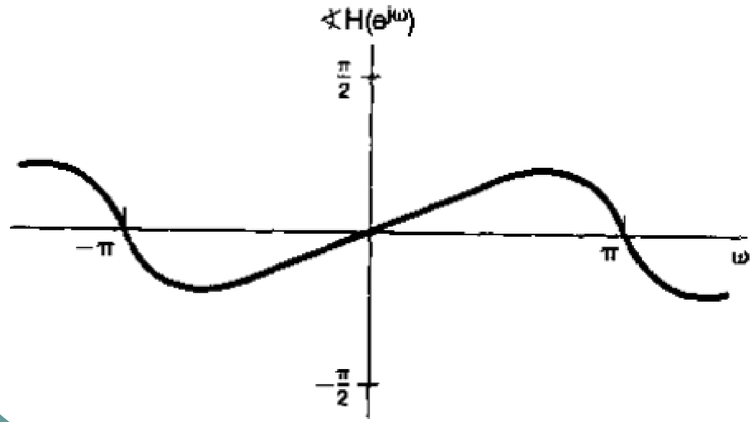




3.10 Examples of continuous-time filters described by differential equations

- When $a = -0.6$





3.11 Examples of discrete-time filters described by differential equations

- General form of nonrecursive discrete-time filters (moving-average filter)

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]$$

- An example:

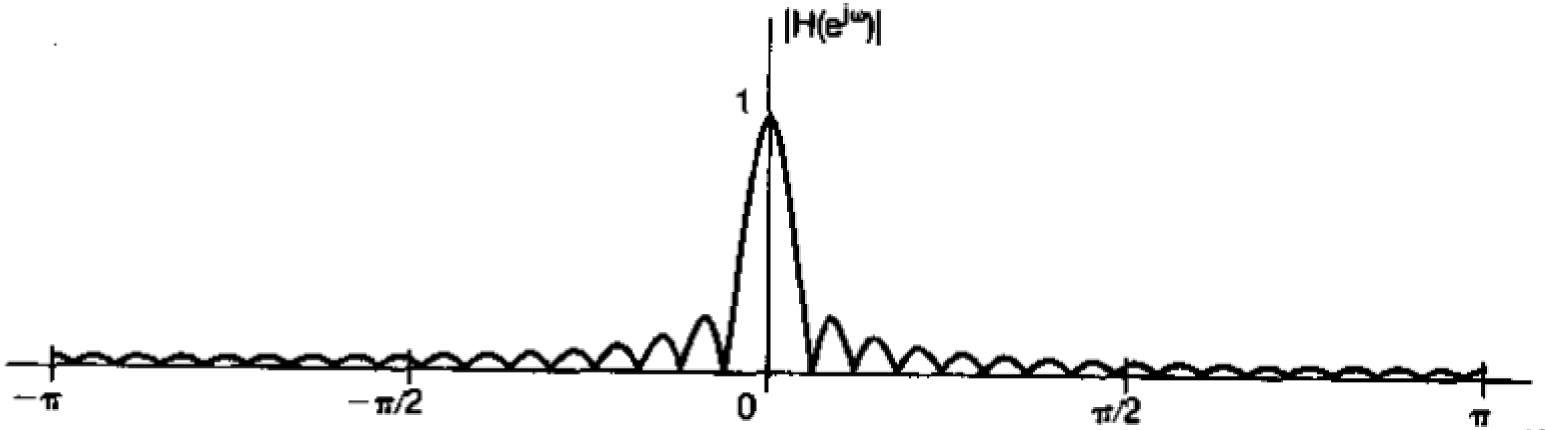
$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]$$

$$\Rightarrow \quad H(e^{j\omega}) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{j\omega k}$$

$$= \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}$$

3.11 Examples of discrete-time filters described by differential equations

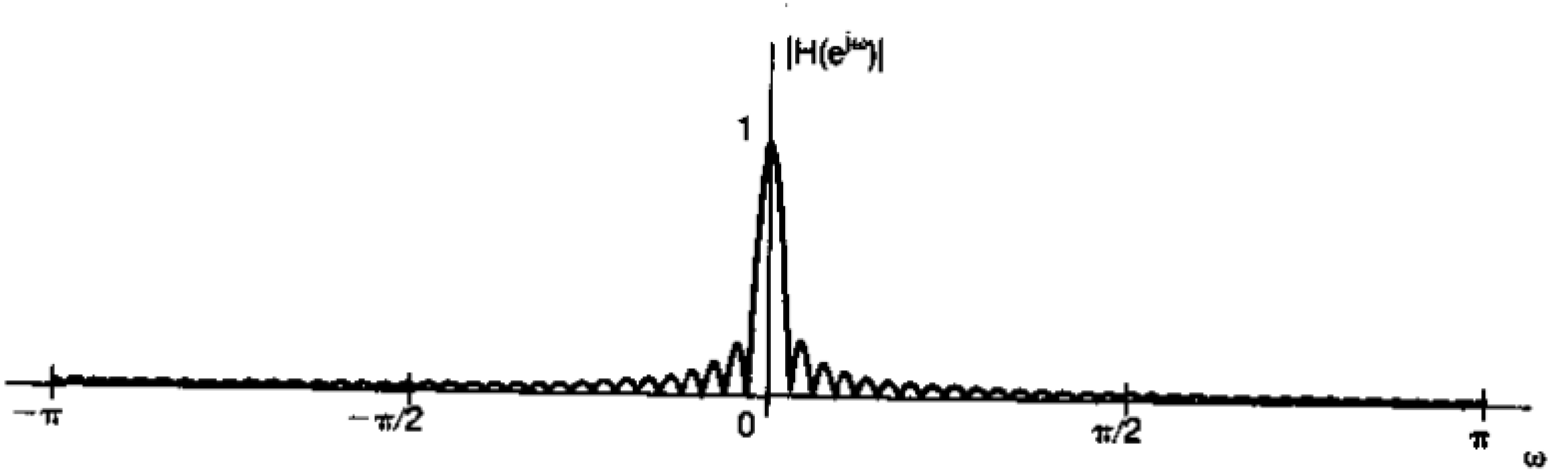
- When M=N=16



(a)

3.11 Examples of discrete-time filters described by differential equations

- When M=N=32



3.11 Examples of discrete-time filters described by differential equations

- Use nonrecursive discrete-time to perform highpass filtering operation

$$y[n] = \frac{x[n] - x[n-1]}{2}$$

$$h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]]$$

$$H(e^{j\omega})\ \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2)$$